Supplemental Information to “Continuous-Time Stochastic Games with Imperfect Public Monitoring”

Benjamin Bernard*

This supplemental information contains details to auxiliary proofs that are relatively straightforward adaptations of the corresponding proofs in Bernard (2024).

H Dynamics of the continuation value

H.1 SDE representation of PPE

Proof of Lemma 4.1. Fix a strategy profile A and abbreviate $W := W(S_0, A)$. Note that W is bounded since it takes values in $\mathcal{V}_0$. For the first direction, it remains to show that W satisfies SDE (5). Let $\mathbb{H}$ denote the filtration generated by $(J^{y,y'})_{(y,y')\in\mathcal{Z}}$ and $\int \sigma(S_t)\,\mathrm{d}Z_t$ and let $\mathbb{M}$ be the filtration generated by all orthogonal information in $\mathbb{F}$. Because $(J^{y,y'})_{(y,y')\in\mathcal{Z}}$ are pairwise orthogonal and orthogonal to $\int \sigma(S_t)\,\mathrm{d}Z_t$, Theorem IV.36 in Protter (2005) implies that the stable subspace generated by $\int \sigma(S_t)\,\mathrm{d}Z_t$ and $(J^{y,y'})_{(y,y')\in\mathcal{Z}}$ is the space of all stochastic integrals with respect to these processes. Therefore, by Corollary 1 to Theorem IV.37 in Protter (2005), any square-integrable $\mathbb{F}$-martingale can be uniquely decomposed into the sum of a square-integrable $\mathbb{M}$-martingale and stochastic integrals with respect to $\int \sigma(S_t)\,\mathrm{d}Z_t$ and $(J^{y,y'})_{(y,y')\in\mathcal{Z}}$. To apply the result, fix a player i, a time $T > 0$, and set $w^i_T := W^i_T - r\int_0^T\big(W^i_t - g^i(S_t, A_t)\big)\,\mathrm{d}t$. The unique martingale representation for $\mathbb{E}\big[w^i_T \,\big|\, \mathcal{F}_t\big]$ implies that there exist an $\mathcal{F}_0$-measurable c^i_T, predictable and square-integrable processes $\big(\beta^i_{t,T}\big)_{0\le t\le T}$ and $\big(\delta^i_{t,T}(y,y')\big)_{0\le t\le T}$ for $(y,y')\in\mathcal{Z}$ and a $(\mathbb{M}, Q^A_T)$-martingale N^i with $N^i_0 = 0$ such that

$$w^i_T = c^i_T + r\int_0^T \beta^i_{t,T}\big(\sigma(S_t)\,\mathrm{d}Z_t - \mu(S_t, A_t)\,\mathrm{d}t\big)$$
$$+ \sum_{(y,y')\in\mathcal{Z}} r\int_0^T \delta^i_{t,T}(y,y')\big(\mathrm{d}J^{y,y'}_t - \lambda_{y,y'}(A_t)\mathrm{d}t\big) + N^i_{T,T}.$$

To prove that (5) holds, it remains to show that c^i_T, $\beta^i_{t,T}$, $\delta^i_{t,T}(y,y')$ and $N^i_{t,T}$ do not depend

*Department of Economics, University of Wisconsin–Madison, bbernard3@wisc.edu.

on T. It follows from (2) and Fubini's theorem that

$$w^i_T = W^i_T + r\int_0^T g^i(S_t, A_t)\,\mathrm{d}t - r\int_0^\infty \int_0^{s\wedge T} r\mathrm{e}^{-r(s-t)}\mathbb{E}_{Q^A_s}\big[g^i(S_s, A_s)\;\big|\;\mathcal{F}_t\big]\,\mathrm{d}t\,\mathrm{d}s. \tag{40}$$

Let $\tilde{T} \le T$ and take conditional expectations on $\mathcal{F}_{\tilde{T}}$ under Q^A_T of (40) to deduce that

$$\begin{aligned}
\mathbb{E}_{Q^A_T}\big[w^i_T\,\big|\,\mathcal{F}_{\tilde{T}}\big] - w^i_{\tilde{T}} &= \mathbb{E}_{Q^A_T}\big[W^i_T\,\big|\,\mathcal{F}_{\tilde{T}}\big] - W^i_{\tilde{T}} + r\int_{\tilde{T}}^T \mathbb{E}_{Q^A_t}\big[g^i(S_t, A_t)\big|\,\mathcal{F}_{\tilde{T}}\big]\,\mathrm{d}t \\
&\qquad - r\int_{\tilde{T}}^\infty \int_{\tilde{T}}^{s\wedge T} r\mathrm{e}^{-r(s-t)}\mathbb{E}_{Q^A_s}\big[g^i(S_s, A_s)\big|\,\mathcal{F}_{\tilde{T}}\big]\,\mathrm{d}t\,\mathrm{d}s \\
&= \mathbb{E}_{Q^A_T}\big[W^i_T\,\big|\,\mathcal{F}_{\tilde{T}}\big] - W^i_{\tilde{T}} - \int_T^\infty r\mathrm{e}^{-r(s-T)}\mathbb{E}_{Q^A_s}\big[g^i(S_s, A_s)\big|\,\mathcal{F}_{\tilde{T}}\big]\,\mathrm{d}s \\
&\qquad + \int_{\tilde{T}}^\infty r\mathrm{e}^{-r(s-\tilde{T})}\mathbb{E}_{Q^A_s}\big[g^i(S_s, A_s)\big|\,\mathcal{F}_{\tilde{T}}\big]\,\mathrm{d}s \\
&= 0.
\end{aligned}$$

Taking $\tilde{T} = 0$, this shows that $c^i_T = W^i_0$ does not depend on T. For arbitrary $\tilde{T}$, it implies

$$\begin{aligned}
w^i_{\tilde{T}} = W^i_0 &+ r\int_0^{\tilde{T}} \beta^i_{t,T}\big(\sigma(S_t)\,\mathrm{d}Z_t - \mu(S_t, A_t)\,\mathrm{d}t\big) \\
&+ \sum_{(y,y')\in\mathcal{Z}} r\int_0^{\tilde{T}} \delta^i_{t,T}(y, y')\big(\mathrm{d}J^{y,y'}_t - \lambda_{y,y'}(A_t)\,\mathrm{d}t\big) + N^i_{\tilde{T},T},
\end{aligned}$$

hence $\beta^i_{\cdot,T} = \beta^i_{\cdot,\tilde{T}}$ and $\delta^i_{\cdot,T}(y, y') = \delta^i_{\cdot,\tilde{T}}(y, y')$ for every $(y, y') \in \mathcal{Z}$ a.e. on $[0, \tilde{T}]$ and $N^i_{\tilde{T},T} = N^i_{\tilde{T},\tilde{T}}$ a.s. by the uniqueness of the orthogonal decomposition. Taking $\mathcal{F}_t$-conditional expectations under Q^A_T, we deduce $N^i_{t,\tilde{T}} = N^i_{t,T}$ a.s. for $t \in [0, \tilde{T}]$, proving that the integral representation is independent of T. We thus omit the subscript T and $\tilde{T}$ of the constructed processes. To arrive at (5), we move the Poisson processes that do not correspond to the current state into the orthogonal part by setting

$$M^i := N^i + \sum_{(y,y')\in\mathcal{Z}} r\int_0^{\cdot} \delta^i_t(y, y')1_{\{S_{t-}\neq y\}}\big(\mathrm{d}J^{y,y'}_t - 1\,\mathrm{d}t\big)$$

and $\delta^i(y) := \delta^i(S_-, y)1_{\{(S_-,y)\in\mathcal{Z}\}}$ for any $y \in \mathcal{Y}$. Because N^i is orthogonal of Z and $(J^{y,y'})_{(y,y')\in\mathcal{Z}}$, and the processes $J^{y,y'}$ are orthogonal to each other and of Z, it follows that M^i is a martingale orthogonal to $\int \sigma(S_t)\,\mathrm{d}Z_t$ and to the processes $J^y := \sum_{0<s\le t} \Delta J^{S_{s-},y}_t$ that counts transitions to y from the current state. Note that $\delta^i(y)$ is square-integrable since the

processes $\delta^i(y, y')$ are, and it is predictable because $\delta^i(y, y')$ and S_- are. By construction,

$$M^i + \sum_{y\in\mathcal{Y}} r \int_0^\cdot \delta^i(y)\big(\mathrm{d}J_t^y - \lambda_{S_{t-},y}(A_t)\,\mathrm{d}t\big) = N^i + \sum_{(y,y')\in\mathcal{Z}} r \int_0^{\tilde{T}} \delta^i_{t,T}(y, y')\big(\mathrm{d}J_t^{y,y'} - \lambda_{y,y'}(A_t)\,\mathrm{d}t\big),$$

hence W satisfies (5) for processes β^i, $\big(\delta^i(y)\big)_{y\in\mathcal{Y}}$, and M^i.

To show the converse, let $(W, S, A, \beta, \delta, Z, (J^y)_y, M)$ satisfy (5). Itō's formula yields

$$\begin{aligned} \mathrm{d}(\mathrm{e}^{-rt}W_t^i) &= -r\mathrm{e}^{-rt}g^i(S_t, A_t)\,\mathrm{d}t + r\mathrm{e}^{-rt}\beta_t^i\left(\sigma(S_t)\,\mathrm{d}Z_t - \mu(S_t, A_t)\,\mathrm{d}t\right) \\ &\quad + r\mathrm{e}^{-rt}\sum_{y\in\mathcal{Y}}\delta_t^i(y)\big(\mathrm{d}J_t^y - \lambda_{S_{t-},y}(A_t)\,\mathrm{d}t\big) + \mathrm{e}^{-rt}\,\mathrm{d}M_t^i. \end{aligned} \tag{41}$$

Since M^i is strongly orthogonal to $\int \sigma(S_t)\,\mathrm{d}Z_t$ and $(J^y)_{y\in\mathcal{Y}}$, it is also strongly orthogonal to the density process given in (22). Therefore, M^i is a martingale also under Q^A. Integrating (41) from t to T and taking Q_T^A-conditional expectations on $\mathcal{F}_t$ thus yields

$$W_t^i = \int_t^T r\mathrm{e}^{-r(s-t)}\mathbb{E}_{Q_s^A}\left[g^i(S_s, A_s) \mid \mathcal{F}_t\right]\mathrm{d}s + \mathrm{e}^{-r(T-t)}\mathbb{E}_{Q_T^A}\left[W_T^i \middle| \mathcal{F}_t\right].$$

Since W is bounded, the second summand converges to zero a.s. as T tends to ∞, hence W_t^i is indeed player i's continuation value under strategy profile A in state S_t. □

Proof of Lemma 4.3. Fix a strategy profile A and let $\tilde{A}$ be a strategy profile involving a unilateral deviation of some player i, that is, $\tilde{A}^{-i} = A^{-i}$ a.e. By Lemma 4.1, there exist processes β, δ, and M that satisfy (5). Integrating (41) from t to u yields

$$\begin{aligned} W_t^i(S_t, A) &= -\int_t^u \mathrm{e}^{-r(s-t)}\Big(\beta_s^i\left(\sigma(S_s)\,\mathrm{d}Z_s - \mu(S_s, A_s)\,\mathrm{d}s\right) - g^i(S_s, A_s)\,\mathrm{d}s - \mathrm{d}M_s^i\Big) \\ &\quad - \sum_{y\in\mathcal{Y}}\int_t^u \mathrm{e}^{-r(s-t)}\delta_s^i(y)\left(\mathrm{d}J_s^y - \lambda_{S_{s-},y}(A_s)\,\mathrm{d}s\right) + \mathrm{e}^{-r(u-t)}W_u^i(S_u, A). \end{aligned}$$

Note that $\mathrm{e}^{-r(u-t)}W_u^i(S_u, A)$ vanishes as $u \to \infty$ because $W(S, A)$ is in the bounded set $\mathcal{V}_0$. Since M is a martingale up to time u also under $Q_u^{\tilde{A}}$, taking conditional expectations yields

$$\begin{aligned} W_t^i\big(S_t, \tilde{A}\big) &= \lim_{u\to\infty}\mathbb{E}_{Q_u^{\tilde{A}}}\left[\int_t^u r\mathrm{e}^{-r(s-t)}g^i\big(S_s, \tilde{A}_s\big)\,\mathrm{d}s\middle|\mathcal{F}_t\right] \\ &= W_t^i(S_t, A) + \lim_{u\to\infty}\mathbb{E}_{Q_u^{\tilde{A}}}\Bigg[\int_t^u r\mathrm{e}^{-r(s-t)}\Big(\big(g^i\big(S_s, \tilde{A}_s\big) - g^i(S_s, A_s)\big)\,\mathrm{d}s \\ &\quad + \beta_s^i\left(\sigma(S_s)\,\mathrm{d}Z_s - \mu(S_s, A_s)\,\mathrm{d}s\right) + \sum_{y\in\mathcal{Y}}\delta_s^i(y)\left(\mathrm{d}J_s^y - \lambda_{S_{s-},y}(A_s)\,\mathrm{d}s\right)\Big)\Bigg|\mathcal{F}_t\Bigg] \text{ a.s.} \end{aligned}$$

Note that the state process S is the same process under strategy profile A and $\tilde{A}$ (as a map from $\Omega \times [0, \infty)$ to $\mathcal{Y}$), it merely has a different distribution under Q_u^A and $Q_u^{\tilde{A}}$. Because β is constructed using a martingale representation result for the bounded random variable w_T^i in (40), the process $\int_t^{\cdot} r\mathrm{e}^{-r(s-t)}\beta_s^i \left(\sigma(S_s)\, \mathrm{d}Z_s - \mu(S_s, A_s)\, \mathrm{d}s\right)$ is a bounded mean oscillation (BMO) martingale under the probability measure Q_u^A up to any time $u > t$. It follows from Theorem 3.6 in Kazamaki (2006) that $\int_t^{\cdot} r\mathrm{e}^{-r(s-t)}\beta_s^i \left(\sigma(S_s)\, \mathrm{d}Z_s - \mu(S_s, \tilde{A}_s)\, \mathrm{d}s\right)$ is a martingale under $Q_u^{\tilde{A}}$. Since $W(S, A)$ lies in $\mathcal{V}_0$, each $\delta(y)$ is uniformly bounded P-a.s., hence also $Q_u^{\tilde{A}}$-a.s. for any $u > t$. The lemma after Theorem IV.29 in Protter (2005) thus implies that $\int_t^{\cdot} r\mathrm{e}^{-r(s-t)}\delta_s^i(y) \left(\mathrm{d}J_s^y - \lambda_{S_{s-},y}(\tilde{A}_s)\, \mathrm{d}s\right)$ is a $Q_u^{\tilde{A}}$-martingale up to any time $u > t$. Together with Fubini's theorem, this implies

$$\begin{aligned} W_t^i(S_t, \tilde{A}) - W_t^i(S_t, A) = \int_t^\infty \mathrm{e}^{-r(s-t)} \mathbb{E}_{Q_s^{\tilde{A}}}\Big[g^i(S_s, \tilde{A}_s) - g^i(S_s, A_s) + \beta_s^i \left(\mu(S_s, \tilde{A}_s)\right. \qquad (42) \\ \left. - \mu(S_s, A_s)\right) + \delta_s^i \left(\lambda(S_s, \tilde{A}_s) - \lambda(S_s, A_s)\right) \Big| \mathcal{F}_t \Big]\, \mathrm{d}s \ \text{ a.s.} \end{aligned}$$

If (β, δ) enforces A in state S, the above conditional expectation is non-positive, hence A is a PPE. To show the converse, assume towards a contradiction that there exist a player i and a set $\Xi \subseteq \Omega \times [0, \infty)$ with $P \otimes Lebesgue(\Xi) > 0$, such that some strategy $\hat{A}^i$ satisfies

$$g^i(S, \hat{A}) - g^i(S, A) + \beta^i \left(\mu(S, \hat{A}) - \mu(S, A)\right) + \delta^i \left(\lambda(S, \hat{A}) - \lambda(S, A)\right) > 0 \qquad (43)$$

on the set Ξ, where we denoted $\hat{A} = \left(\hat{A}^i, A^{-i}\right)$ for the sake of brevity. Because β and δ are predictable and both A and $\hat{A}$ are the limit of a sequence of predictable processes $A^{(n)}$ and $\hat{A}^{(n)}$, there must exist predictable $\Xi_n \to \Xi$ with $P \otimes Lebesgue(\Xi_n) > 0$ such that (43) holds for processes $A^{(n)}$ and $\hat{A}^{(n)}$. Thus, $\tilde{A}^i := \lim_{n\to\infty} \hat{A}^{(n),i} 1_{\Xi_n} + A^{(n),i} 1_{\Xi_n^c}$ is a valid strategy for player i. For $\tilde{A} = \left(\tilde{A}^i, A^{-i}\right)$, the expectation in (42) is strictly positive for $t = 0$, a contradiction to the fact that A is a PPE. $\square$

H.2 Concatenations of solutions to SDE (5)

In many of the proofs, we will concatenate enforceable solutions to SDE (5), which is subject to some subtle measurability issues. Without restrictions on β and δ, solutions to (5) are weak solutions, which means that the entire *stochastic framework* $(\Omega, \mathcal{F}, \mathbb{F}, P, Z, (J^{y,y'})_{(y,y')\in\mathcal{Z}})$ is part of the solution. Thus, more formally, the set $\mathcal{B}_y(\mathcal{W})$ is the set of all payoffs $w \in \mathcal{V}$, for which there exists a stochastic framework $(\Omega, \mathcal{F}, \mathbb{F}, P, Z, (J^{y,y'})_{(y,y')\in\mathcal{Z}})$ containing a solution

$(W, S, A, \beta, \delta, M)$ to (5) on $[\![0, \tau_1]\!]$ with $W_0 = w$, $S_0 = y$, and $W_{\tau_1} \in \mathcal{W}_{S_{\tau_1}}$ P-a.s., such that on $[\![0, \tau_1[\![$, $W \in \mathcal{B}(\mathcal{W})$ and (β, δ) enforces A, where τ_1 is the first jump time of any of the processes $(J^{y,y'})_{(y,y')\in\mathcal{Z}}$.[17] At time τ_1, we would like to attain W_{τ_1} with another enforceable solution to (5) that exists by definition of $\mathcal{B}_{S_{\tau_1}}(\mathcal{W})$. However, the continuation solutions may live on separate probability spaces for each realization of W_{τ_1}. The following result, adapted from Lemma A.1 in Bernard (2024), establishes that there exists a probability space that contains the concatenation.

Lemma H.1. *Let X be a $\mathcal{V}_0$-valued random variable with distribution ν. Equivalent are:*

(i) $X \in \mathcal{B}_y(\mathcal{W})$ *ν-a.s.,*

(ii) There exists $\left(\Omega, \mathcal{F}, \mathbb{F}, P, Z, (J^{y,y'})_{(y,y')\in\mathcal{Z}}\right)$ *containing a solution* $(W, S, A, \beta, \delta, M)$ *to* (5) *such that X is $\mathcal{F}_0$-measurable with $\nu = P \circ X^{-1}$, $W_0 = X$ P-a.s., $S_0 = y$ P-a.s., $W_{\tau_1} \in \mathcal{W}_{S_{\tau_1}}$ P-a.s., and on almost everywhere on $[\![0, \tau_1[\![$, (β, δ) enforces A and $W \in \mathcal{B}_y(\mathcal{W})$, where τ_1 is the first jump time of any of the processes $(J^{y,y'})_{(y,y')\in\mathcal{Z}}$.*

Moreover, a similar equivalence holds for random variables $X \in \mathcal{E}(y)$ ν-a.s., where the solution in (ii) simply has to satisfy $W_0 = X$ P-a.s., $S_0 = y$ P-a.s., and (β, δ) enforces A.

Crucial in the proof are (a) that the path space of the solutions coincide for different realizations of X and (b) that the path space is complete and separable for the Skorohod metric. Then the different probability spaces can be aggregated with a regular conditional probability. Neither of these properties depend on whether we are studying pure or behavior strategies since in either case the path space of all strategy profiles is compact by Tychonov's theorem. Since neither property depends on whether we study repeated or stochastic games, the proof of Lemma A.1 in Bernard (2024) applies directly. We can now formalize the concatenation procedure as follows.

Lemma H.2. *Fix a state y, a bounded payoff set $\mathcal{X}_y$, and a family of payoff sets $\mathcal{W} = (\mathcal{W}_{y'})_{y'\in\mathcal{Y}}$ such that for every $w \in \mathcal{X}_y$, there exists a stochastic framework* $\left(\Omega, \mathcal{F}, \mathbb{F}, P, Z, (J^{y,y'})_{(y,y')\in\mathcal{Z}}\right)$ *containing a solution $(W, S, A, \beta, \delta, M)$ to* (5) *for S defined in* (21) *on the stochastic interval $[\![0, \tau\wedge\tau_1]\!]$, where τ_1 is the first jump time of $(J^{y,y'})_{(y,y')\in\mathcal{Z}}$ and τ is an $\mathbb{F}$-stopping time such that:*

(i) $W_0 = w$ *and* $S_0 = y$ *P-a.s.,*

[17] We say that a stochastic process X satisfies a certain property on a set $\Xi \subseteq \Omega \times [0, \infty)$ if $X_t(\omega)$ satisfies that property for $P \otimes Lebesgue$-almost every $(\omega, t) \in \Xi$.

(ii) $W \in \mathcal{X}_y$, (β, δ) *enforces* A *in* S, *and* $W + r\delta(y') \in \mathcal{W}_{y'}$ *for each* $y' \in \mathcal{Y}$ *on* $[\![0, \tau \wedge \tau_1)\!)$,

(iii) *On* $\{\tau < \tau_1\}$, *we have* $W_\tau \in \mathcal{B}_y(\mathcal{W})$.

Then for every $w \in \mathcal{X}_y$, *there exists a stochastic framework* $\big(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{P}, \hat{Z}, (\hat{J}^{y,y'})_{y,y' \in \mathcal{Z}}\big)$ *containing a solution* $(\hat{W}, \hat{S}, \hat{A}, \hat{\beta}, \hat{\delta}, \hat{M})$ *to* (5) *that coincides with* $(W, S, A, \beta, \delta, M)$ *on* $[\![0, \tau \wedge \tau_1)\!)$, *such that* $(\hat{\beta}, \hat{\delta})$ *enforces* $\hat{A}$ *in state* $\hat{S}$ *on* $[\![0, \hat{\tau}_1)\!)$, $\hat{W} \in \mathcal{B}_{\hat{S}}(\mathcal{W})$ *on* $[\![\tau, \hat{\tau}_1)\!)$ *and* $\hat{W}_{\hat{\tau}_1} \in \mathcal{W}_{\hat{S}_{\hat{\tau}_1}}$ $\hat{P}$*-a.s., where* $\hat{\tau}_1$ *is the first jump time of any of the processes* $(\hat{J}^{y,y'})_{y,y' \in \mathcal{Z}}$. *In particular,* $\mathcal{X}_y \subseteq \mathcal{B}_y(\mathcal{W})$.

Moreover, if $\mathcal{W}_{y'} \subseteq \mathcal{B}_{y'}(\mathcal{W})$ *for each* $y' \in \mathcal{Y}$, *then the above result holds for the first stopping time* $\hat{\tau}_1$ *of any of the processes* $(\hat{J}^{y,y'})_{y,y' \in \mathcal{Z}}$ *strictly after* τ_1.

Proof. Fix $w \in \mathcal{X}_y$, a stochastic framework $\big(\Omega, \mathcal{F}, \mathbb{F}, P, Z, (J^{y,y'})_{(y,y') \in \mathcal{Z}}\big)$, an $\mathbb{F}$-stopping time τ, and a solution $(W, S, A, \beta, \delta, M)$ to (5) on $[\![0, \tau]\!]$ with all the stated properties. By Lemma H.1, on $\{\tau < \tau_1\}$ there exists a stochastic framework $\big(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P}, \tilde{Z}, (\tilde{J}^{y,y'})_{(y,y') \in \mathcal{Z}}\big)$ containing a $\mathcal{W}$-enforceable solution $\big(\tilde{W}, \tilde{S}, \tilde{A}, \tilde{\beta}, \tilde{\delta}, \tilde{M}\big)$ to (5) such that $\tilde{W}_0$ is distributed under $\tilde{P}$ as W_τ is under P. We choose Ω and $\tilde{\Omega}$ to be the Polish path spaces of these solutions so that there exists a regular conditional probability $\hat{P}$ on $\hat{\Omega} = \Omega \times \tilde{\Omega}$ with marginal P on Ω. Let $\hat{\mathbb{F}} := \big(\hat{\mathcal{F}}_t\big)_{t \geq 0}$ denote the $\hat{P}$-augmented filtration such that $\hat{\mathcal{F}}_t$ contains $\mathcal{F}_{t \wedge \tau} \vee \tilde{\mathcal{F}}_{t - \tau \vee 0}$. Define the processes $\hat{Z}$ and $\hat{J}^{y,y'}$ by setting

$$\hat{Z}_t = Z_{t \wedge \tau} + \tilde{Z}_{t-\tau} 1_{\{t > \tau\}}, \qquad \hat{J}^{y,y'}_t = J^{y,y'}_{t \wedge \tau} + \tilde{J}^{y,y'}_{t-\tau} 1_{\{t > \tau\}}.$$

Because Brownian motion and Poisson processes have independent and identically distributed increments, $\hat{Z}$ is an $\hat{\mathbb{F}}$-Brownian motion and $\hat{J}^{y,y'}$ for any $(y, y') \in \mathcal{Z}$ is an $\hat{\mathbb{F}}$-Poisson process. Define the concatenated processes $\hat{W}$, $\hat{S}$, $\hat{A}$, $\hat{\beta}$, $\hat{\delta}$, and $\hat{M}$ by setting

$$\hat{W} := \big(W 1_{[\![0,\tau]\!]} + \tilde{W}_{\cdot - \tau} 1_{(\!(\tau, \infty)\!)}\big) 1_{\{\tau < \tau_1\}} + W 1_{\{\tau \geq \tau_1\}}$$

and similarly for $\hat{A}$, $\hat{S}$, $\hat{\beta}$, $\hat{\delta}$, and $\hat{M}$. By construction, $\hat{S}$ satisfies (21) for $\hat{A}$ and $\big(\hat{W}, \hat{S}, \hat{A}, \hat{\beta}, \hat{\delta}, \hat{M}\big)$ coincides with $(W, S, A, \beta, \delta, M)$ on $[\![0, \tau \wedge \tau_1)\!)$. Since $\hat{P}$ is a regular conditional probability, $\hat{W}_0 = w$ and $\hat{W}_{\hat{\tau}_1} \in \mathcal{W}_{\hat{S}_{\hat{\tau}_1}}$ $\hat{P}$-a.s. and $\big(\hat{W}, \hat{A}, \hat{\beta}, \hat{\delta}, \hat{M}\big)$ has the desired properties on $[\![0, \hat{\tau}_1)\!)$.

The above concatenation is carried out only on the event $\{\tau < \tau_1\}$. If $\mathcal{W}_{y'} \subseteq \mathcal{B}_{y'}(\mathcal{W})$ holds for each state y', then (ii) implies that $W_{\tau_1} \in \mathcal{B}_{S_{\tau_1}}(\mathcal{W})$ on $\{\tau_1 \leq \tau\}$. We can thus carry out the same steps on $\{\tau_1 \leq \tau\}$ attaining W_{τ_1} to obtain a solution until the first state transitionjump time $\hat{\tau}_1$ strictly after τ_1. □

Proof of Lemma 4.7. Fix any family $\mathcal{W}$ of bounded self-generating sets and any state y. Since $\mathcal{B}(\mathcal{W})$ is the largest family of bounded payoff sets generated by $\mathcal{W}$, we must have $\mathcal{W}_{y'} \subseteq \mathcal{B}_{y'}(\mathcal{W})$ for any state y'. Fix now an arbitrary $w \in \mathcal{W}_{y'}$ and any enforceable solution to (5) on $[\![0, \tau_1]\!]$ attaining $w \in \mathcal{W}_{y'}$. An application of Lemma H.2 for $\mathcal{X}_y = \mathcal{W}_y$ and $\tau = \tau_1$ shows that the solution can be extended to the second state transition τ_2. Since there are countably many state transitions, an iteration of this procedure shows that w is attainable by an enforceable solution on $\Omega \times [0, \infty)$, hence $w \in \mathcal{E}(y)$ by Lemma 4.3.

To show that $\mathcal{E}$ is self-generating, fix any y and $w \in \mathcal{E}(y)$. By Lemma 4.3, there exists an enforceable solution $(W, A, S, \beta, \delta, M)$ to (5) for S defined in (21) with $W_0 = w$ a.s. Fix an arbitrary stopping time τ and let ν denote the distribution of W_τ. Since the continuation solution after any stopping time τ satisfies condition (ii) of Lemma H.1, that lemma implies $W_\tau \in \mathcal{E}(S_\tau)$ a.s. Because τ was arbitrary, this shows that $\mathcal{E}$ is self-generating. □

H.3 Proof of Proposition 4.9

Proof of Proposition 4.9. As in the proof of Proposition 6.8, $\mathcal{B}_y(\mathcal{V}_0^*) \subseteq \mathcal{V}_0^*$ and each sequence $(\mathcal{W}_y^n)_{n\geq 0}$ is decreasing in the set-inclusion sense with $\mathcal{W}_y^n \supseteq \mathcal{E}(y)$ because $\mathcal{B}$ is monotone. It remains to show that $\mathcal{W}_y^\infty \subseteq \mathcal{B}_y(\mathcal{W}^\infty)$ for each state y so that $\mathcal{W}^\infty$ is self-generating and, hence, $\mathcal{W}_y^\infty = \mathcal{E}(y)$. Fix an arbitrary payoff $w \in \mathcal{W}_y^\infty$. Since $w \in \mathcal{W}_y^n$ for each n, there exist enforceable solutions $(W^n, S^n, A^n, \beta^n, \delta^n, M^n)$ to (5) on some stochastic basis $\big(\Omega^n, \mathcal{F}^n, \mathbb{F}^n, Z^n, P^n, (J^{n,y,y'})_{(y,y')\in\mathcal{Z}}\big)$ with $W^n \in \mathcal{W}_y^n$ a.e. on $[\![0, \tau_1)\!)$ and $W_{\tau_1}^n \in \mathcal{W}_{S_{\tau_1}}^{n-1}$ a.s. We will show that the solutions converge in law along a subsequence to a limit $(W, S, A, \beta, \delta, M)$ that solves (5). We accomplish this by showing that the sequences of stochastic processes are tight so that convergence follows from Prokhorov's theorem.[18]

The first step to establish tightness is to show that the quadratic variation of these processes is uniformly bounded. Because the optimality equation (9) is Lipschitz continuous, the maximizer $\beta^*(w, N)$ is uniformly bounded on the compact set $\mathcal{V}_0^* \times S^1$. It is thus sufficient to show that we can choose public randomization M^n with finite variation so that W^n remains on the boundary of $\mathcal{W}_y^n$. First, we can attain w in the interior of $\mathcal{W}_y^n$ with public randomization M_0^n at time 0 that takes values in $\partial\mathcal{W}_y^n$. Once on the boundary, the continuation value can enter the interior of $\mathcal{W}_y^n$ only at payoff pairs in $\mathcal{K}_y(\mathcal{W}^{n-1})$ with strictly inward drift. Fix such a payoff pair w_0 and (α_0, δ_0) such that the drift rate points strictly towards the interior of

[18] See, for example, Theorem VI.3.5 in Jacod and Shiryaev (2002).

$\mathcal{W}_y^n$. The straight line through $g(y, \alpha_0) + \delta_0 \lambda(y, \alpha_0)$ and w_0 intersects $\partial \mathcal{W}_y^n$ at w_0 and some other payoff pair v. Let $\hat{J}^n$ be a Poisson process orthogonal to Z^n and $(J^{n,y,y'})_{(y,y') \in \mathcal{Z}}$ with instantaneous intensity $r \| w_0 - g(y, \alpha_0) - \delta_0 \lambda(y, \alpha_0) \| / \| v - w_0 \|$ and set

$$\mathrm{d}\hat{M}_t^n = (v - w_0)\, \mathrm{d}\hat{J}_t^n - r\big(w_0 - g(y, \alpha_0) - \delta_0 \lambda(y, a_0)\big)\, \mathrm{d}t.$$

By construction, $\hat{M}^n$ is a martingale and the solution to (5) with $A \equiv \alpha_0$, $\beta \equiv 0$, $\delta \equiv \delta_0$, and $\hat{M}^n$ has zero drift. It remains at w_0 until either a jump of $\hat{J}^n$ occurs, at which point it jumps to $v \in \partial \mathcal{W}_y^n$, or a jump of $J^{n,y,y'}$ occurs, at which point the process jumps to $\mathcal{W}_{y'}^{n-1}$. A countable concatenation of this procedure yields a solution that remains on $\partial \mathcal{W}_y^n$ until τ_1. We deduce that we can choose M^n with finite variation and β^n and such that $\max_y \| \beta^n \sigma(y) \| \leq K$ for some constant $K > 0$.

The next step is to verify that $(W^n)_{n \geq 0}$ is tight by verifying that the associated semimartingale characteristics satisfy the sufficient conditions of Theorem VI.5.17 in Jacod and Shiryaev (2002). We verify those conditions for $\tilde{W}^n := (e^{-rt} W_t^n)_{t \geq 0}$ because its semimartingale characteristics or more easily computed. Convergence of $(\tilde{W}^n)_{n \geq 0}$ in law by Prokhorov's theorem then implies convergence in law of $(W^n)_{n \geq 0}$. Since each $\tilde{W}^n$ is uniformly bounded under Q_{A^n}, it is a so-called special semimartingale, admitting a unique decomposition $\tilde{W}^n = W_0^n + B^n + \tilde{M}^n$ into a predictable process B^n of finite variation and a Q_{A^n}-local martingale $\tilde{M}^n$. The semimartingale characteristics of $\tilde{W}^n$ under Q_{A^n} is the triplet (B^n, C^n, ν^n), where C^n is the predictable quadratic variation of $\tilde{M}^n$ and ν^n is the compensated jump measure of $\tilde{W}^n$; see Chapter II of Jacod and Shiryaev (2002) for a detailed introduction.[19] It follows from (41) that the first two characteristics of $\tilde{W}^n$ are given by

$$B^n = -\int_0^{\cdot} r \mathrm{e}^{-rt} g(y, A_t^n)\, \mathrm{d}t, \qquad C^{n,ij} = \int_0^{\cdot} r^2 \mathrm{e}^{-2rt} \sigma(y)^\top \beta_t^{n,i\top} \beta_t^{n,j} \sigma(y)\, \mathrm{d}t.$$

Define the auxiliary processes $G^n := \mathrm{Var}\big(B^{n,1}\big) + \mathrm{Var}\big(B^{n,2}\big) + C^{n,11} + C^{n,22}$ for each n. Since we chose $\beta^n \sigma(y)$ uniformly bounded by K, it follows that each G^n is majorized by[20]

$$F := \int_0^{\cdot} r \mathrm{e}^{-rt} \max_{y \in \mathcal{Y}} \max_{a \in \mathcal{A}} \big| g^1(y, a) + g^2(y, a) \big|\, \mathrm{d}t + 2 \int_0^{\cdot} r^2 \mathrm{e}^{-2rt} K^2\, \mathrm{d}t.$$

[19]Because each $\tilde{W}^n$ is bounded, we do not need to truncate its jumps to compute the semimartingale characteristics. In the notation of Jacod and Shiryaev (2002), we can choose "truncation function" $h(x) = x$, hence the second modified characteristic coincides with the second characteristic.

[20]A process F majorizes G if $F - G$ is increasing.

Aldous' criterion implies that $(G^n)_{n\geq 0}$ is tight; see Theorem VI.4.5 in Jacod and Shiryaev (2002). Thus, the sequence $(G^n)_{n\geq 0}$ converges in law along a subsequence to some limit process G by Prokhorov's theorem. Since F is deterministic, F majorizes G. Together with the fact that $\left(\tilde{W}_0^n\right)_{n\geq 0} = (M_0^n)_{n\geq 0}$ is tight because each M_0^n takes values in the compact set $\mathcal{V}$, this shows that $\left(\tilde{W}^n\right)_{n\geq 0}$ is tight by virtue of Theorem VI.5.17 in Jacod and Shiryaev (2002).

With the same argument and majorizing process F, it follows that $(B^n)_{n\geq 0}$ and $\left(\tilde{M}^n\right)_{n\geq 0}$ are tight. Since the path space $\mathcal{A}^{[0,\infty)}$ of any A^n is compact by Tychonov's theorem, the sequence $(A^n)_{n\geq 0}$ is uniformly tight. By Prokhorov's theorem, there exists a subsequence $(n_k)_{k\geq 0}$, along which $(\tilde{W}^n)_{n\geq 0}$, $(A^n)_{n\geq 0}$, $(B^n)_{n\geq 0}$, and $(\tilde{M}^n)_{n\geq 0}$ all converge to limits $\tilde{W}$, A, B, and $\tilde{M}$, respectively. Convergence in law implies that $-\int_0^{\cdot} r\mathrm{e}^{-rt} g(y, A_t)\,\mathrm{d}t = \tilde{B}$. After a suitable transformation with Girsanov's theorem, the laws of $(Z^n)_{n\geq 0}$ and $(J^{y,y',n})_{n\geq 0}$ are constant, hence they converge trivially to a Brownian motion Z and Poisson processes $J^{y,y'}$ along $(n_k)_{k\geq 0}$. By Proposition IX.1.1 in Jacod and Shiryaev (2002), the limit process $\tilde{M}$ of $\left(\tilde{M}^n\right)_{n\geq 0}$ is an $(\mathbb{F}, Q^A)$-martingale. Let β and δ be defined by the martingale representation of $\tilde{M}$. By Theorem VI.6.26 in Jacod and Shiryaev (2002), $\left[\tilde{M}^{n_k}, \tilde{M}^{n_k}\right] \to \left[\tilde{M}, \tilde{M}\right]$ in law, showing that $\beta^{n_k} \to \beta$ and $\delta^{n_k} \to \delta$ in law as well.

Convergence in law implies that w can be attained by an enforceable solution to (5) with $W_{\tau_1} \in \mathcal{W}_{S_{\tau_1}}$ a.s., where S is defined from the limiting processes $(J^{y,y'})_{(y,y')\in\mathcal{Z}}$ as in (21). Finally, similarly to the proof of Proposition 6.8, for each n, the solutions W^m remain in $\mathcal{W}_y^m \subseteq \mathcal{W}_y^n$ a.e. on $[\![0, \tau_1)\!)$ for all $m \geq n$. Convergence in law thus implies that on $[\![0, \tau_1)\!)$, we have $W \in \mathcal{W}_y^n$ for all n and, hence, $W \in \mathcal{W}_y^\infty$. Thus, $\mathcal{W}_y^\infty \subseteq \mathcal{B}_y(\mathcal{W}^\infty)$ by maximality of $\mathcal{B}_y(\mathcal{W}^\infty)$. □

I Characterization of $\mathcal{B}(\mathcal{W})$

In this appendix we formalize the heuristic argument of Section 5.1. The first result generalizes Lemma 5.4 shows how to construct enforceable solutions to the SDE (5) that locally remain on a curve that that solve the optimality equations. It is a generalization of Lemma 5.4 in the main text to concave Lipschitz expansions suitable for the perturbation argument. We state the result for general measurable selectors α_*, β_*, and δ_* so that we can apply it to the maximizers in either (8) or (9).

Lemma I.1. *Fix a family $\mathcal{W} = (\mathcal{W}_y)_{y\in\mathcal{Y}}$ of compact and convex payoff sets with Lipschitz expansion $\mathcal{L}$ of $\mathcal{W}$ as defined in Appendix F. Let $\mathcal{C}$ be a continuously differentiable curve oriented by $w \mapsto N_w$ and let α_*, β_*, δ_* be measurable selectors on $\mathcal{C}$ such that for any $w \in \mathcal{C}$:*

(i) $\big(\beta_*(w), \delta_*(w)\big) \in \Xi_{y,\alpha_*(w)}\big(w, N_w, \mathcal{L}\big)$,

(ii) The curvature $\kappa_y(w)$ *of* $\mathcal{C}$ *at* w *satisfies*

$$\kappa_y(w)\big\|T_w{}^\top\beta_*(w)\sigma(y)\big\|^2 = \frac{2}{r}N_w{}^\top\big(g(y, \alpha_*(w)) + \delta_*(w)\lambda(y, \alpha_*(w)) - w\big).$$

(iii) If $\beta_*(w) = 0$ *for a segment of positive length, then* $T_w{}^\top\big(g(y, \alpha_*(w)) + \delta_*(w)\lambda(y, \alpha_*(w)) - w\big)$ *is either non-positive or non-negative throughout the segment.*

Then the solution $(W, S, A, \beta, \delta, M)$ *to* (5) *with* $S_0 = y$*,* $A = \alpha_*(W_-)$*,* $\beta = \beta_*(W_-)$*,* $\delta = \delta_*(W_-)$*, and* $M \equiv 0$ *remains on* $\mathcal{C}$ *until an end point of* $\mathcal{C}$ *is reached or a state change occurs.*

Statement (iii) imposes that the direction of the drift does not change along solutions to the state-transition optimality equation (9). If the direction of the drift changes at some payoff pair w, then $w \in \bar{\mathcal{S}}_y(\mathcal{W})$, hence $w \in \mathcal{B}_y(\mathcal{W})$ by Lemma 7.4 below.

Proof. Fix w in the relative interior of $\mathcal{C}$ and choose $\eta > 0$ small enough such that $N_w{}^\top N_v > 0$ for all $v \in \mathcal{C} \cap B_\eta(w)$, where $B_\eta(w)$ denotes the closed ball around w with radius η. On $B_\eta(w)$, $\mathcal{C}$ admits a local parametrization f in the direction N_w. For any $v \in B_\eta(w)$, define the orthogonal projection $\hat{v} = T_w{}^\top v$ onto the tangent, where T_w is the vector obtained by rotating N_w by 90° in clockwise direction and denote by $\pi(v) = \big(\hat{v}, f(\hat{v})\big)$ the projection of $v \in B_\eta(w)$ onto $\mathcal{C}$ in the direction N_w. Let $\big(W, A, \beta, \delta, Z, (J^{y,y'})_{(y,y')\in\mathcal{Z}}, M\big)$ be a weak solution to (5) starting at $W_0 = w$ with $A = a_*\big(\pi(W_-)\big)$, $\beta = \beta_*\big(\pi(W_-)\big)$, $\delta = \delta_*\big(\pi(W_-)\big)$, and $M \equiv 0$ on $[\![0, \rho)\!)$, where we set $\rho := \tau_1 \wedge \inf\big\{t \geq 0 \,\big|\, W_t \notin B_\eta(w)\big\}$, where τ_1 is the first jump time of any of the processes $(J^{y,y'})_{(y,y')\in\mathcal{Z}}$. By (iii), such a solution exists. Since π is measurable, the processes A, β and δ are all predictable.

We measure the distance of W to $\mathcal{C}$ by $D_t = N_w{}^\top W_t - f\big(\hat{W}_t\big)$. Note that f is differentiable by assumption and $\big(-f'(\hat{W}_t), 1\big) = \ell_t N_t$, where $\ell_t := \big\|\big(-f'(\hat{W}_t), 1\big)\big\|$. Since f is locally convex it is second-order differentiable at almost every point by Alexandrov's Theorem. In particular, f' has Radon-Nikodým derivative $f''(\hat{W}_t) = -\kappa_y(\pi(W_t))\ell_t^3$. It follows from Itō's formula that

$$\begin{aligned}\mathrm{d}D_t &= r\ell_t N_t^\top\big(W_t - g(y, A_t) - \delta_t\lambda(y, A_t)\big)\,\mathrm{d}t + r\ell_t N_t^\top\beta_t\,\big(\sigma(y)\,\mathrm{d}Z_t - \mu(y, A_t)\,\mathrm{d}t\big)\\ &\quad + r\ell_t \sum_{(y,y')\in\mathcal{Z}} N_{t-}{}^\top\delta_t(y')\,\mathrm{d}J_t^{y,y'} - \frac{1}{2}f''(\hat{W}_{t-})\,\mathrm{d}\big[\hat{W}\big]_t,\end{aligned}$$

where we abbreviated $N_t = N_{\pi(W_t)}$ and $T_t = T_{\pi(W_t)}$. Since $N^\top\beta = 0$, the volatility term vanishes and we can write $\beta = TT^\top\beta$. Before the first state transition, we have $\Delta J^{y,y'} \equiv 0$ for

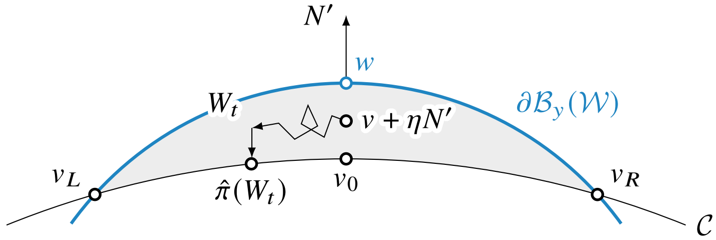

Figure 20: A solution $\mathcal{C}$ to the optimality equation that "cuts through" $\mathcal{B}_y(\mathcal{W})$. The proof of Lemma I.3 compares the law of motion of W attaining $v+\eta N'$ to the maximal incentives in the optimality equation at its projection $\hat{\pi}(W)$ onto $\mathcal{C}$ to deduce that W must escape $\mathcal{B}_y(\mathcal{W})$ with positive probability.

any $(y, y') \in \mathcal{Z}$, hence $[\hat{W}] = \langle \hat{W} \rangle = \langle T_w{}^\top W \rangle$. Using (ii) and the fact that $N_w{}^\top N_t = T_w{}^\top T_t = \ell_t^{-1}$, we obtain that on $[\![0, \rho)\!)$,

$$\begin{aligned} \mathrm{d}D_t &= r\ell_t N_t{}^\top\big(W_t - g(y, A_t) - \delta_t\lambda(y, A_t)\big)\,\mathrm{d}t + \frac{r^2}{2}\kappa_y\big(\pi(W_t)\big)\,\ell_t^3\,\big|T_w{}^\top T_t\big|^2 \big\|T_t{}^\top\beta_t\sigma(y)\big\|^2\,\mathrm{d}t \\ &= rD_t\,\mathrm{d}t, \end{aligned}$$

where we used $N_t{}^\top\big(W_t - \pi(W_t)\big) = N_t{}^\top N_w D_t = \ell_t^{-1} D_t$ in the second equality. It follows that $D_t = D_0 \mathrm{e}^{rt} = 0$ since $D_0 = 0$. On $\{\rho < \tau_1\}$, we can repeat this procedure and concatenate the solutions as in the proof of Lemma H.2 to obtain a solution $(W, S, A, \beta, \delta, M)$ to (5) with $W \in \mathcal{C}$ until time $\tau_1 \wedge \xi$, where ξ is the first time W reaches an end point of $\mathcal{C}$. □

Proof of Lemma 5.4. This is a direct consequence of Lemma I.1 and Lemma 7.4 below. □

I.1 Incentives provided through the public signal

This appendix characterizes the boundary of $\mathcal{B}_y(\mathcal{W})$ where incentives through the public signal are used. To state the result efficiently, let $\mathcal{N}_{\mathcal{X}} := \{(w, N) \mid w \in \partial\mathcal{X} \text{ and } N \in \mathcal{N}_w(\mathcal{X})\}$ denote the *normal bundle* of a convex set $\mathcal{X}$ at some boundary point $w \in \partial\mathcal{X}$.

Lemma I.2. *If* $(w, N) \in \mathcal{N}_{\mathcal{B}_y(\mathcal{W})} \setminus \Gamma_y(\mathcal{W})$, *then* $\partial\mathcal{B}_y(\mathcal{W})$ *is locally a smooth solution to* (9).

The proof of Lemma I.2 proceeds by showing that solutions to (9) with initial conditions $(w, N) \in \mathcal{N}_{\mathcal{B}_y(\mathcal{W})}$ can neither escape nor fall into the interior of $\mathcal{B}_y(\mathcal{W})$. We first show that a solution cannot escape—otherwise the solution for slightly rotated initial conditions would cut through $\mathcal{B}_y(\mathcal{W})$ as in Figure 20; an impossibility by the following lemma.

Lemma I.3. *Let* $w \in \partial\mathcal{B}(\mathcal{W})$ *with outward normal* N'. *Let* $\pi : U \to \partial\mathcal{B}(\mathcal{W})$ *be the projection of a neighborhood* U *of* w *onto* $\partial\mathcal{B}(\mathcal{W})$ *in the direction of* N' *and define the*

Lipschitz expansion $\mathcal{L}$ of $\mathcal{W}$ with $h(v) = \|\pi(v) - v\| 1_{\{v \in \operatorname{int} \mathcal{B}_y(\mathcal{W})\}}$. Let $\mathcal{C}$ be a C^1-solution to (34)*, oriented by $v \mapsto N_v$ with end points $v_L, v_R \in U$. It is impossible that the following properties hold simultaneously:*

(i) $v_L + \varepsilon N' \notin \mathcal{B}_y(\mathcal{W})$ *and* $v_R + \varepsilon N' \notin \mathcal{B}_y(\mathcal{W})$ *for any* $\varepsilon > 0$*,*

(ii) there exists $v_0 \in \mathcal{C}$ *such that* $v_0 + \eta N' \in \mathcal{B}_y(\mathcal{W})$ *for some* $\eta > 0$*,*

(iii) $\inf_{v \in \mathcal{C}} N_v^\top N' > 0$*,*

(iv) $\inf_{v \in \mathcal{C}} \left| N_v^i \right| > 0$ *for* $i = 1, 2$*,*

(v) $\mathcal{N}_{\mathcal{C}} \cap \Gamma(\mathcal{L}) = \emptyset$.

Because the optimality equation maximizes the curvature over all restricted-enforceable action profiles, any enforceable solution to (5) that attains a payoff pair in $\mathcal{B}_y(\mathcal{W})$ "above the curve" (in the shaded area of Figure 20) must stay above $\mathcal{C}$ with positive probability. Because incentives through the public signal are necessary by (v), the continuation value has a positive diffusion term and will escape on either side with positive probability.

Proof. The proof is lengthy but because the state y is fixed, it is entirely analogous to the proof of Lemma C.2 in Bernard (2024). The only additional observation required is that convexity of $\mathcal{B}_y(\mathcal{W})$ implies that h is concave. Thus, local Lipschitz continuity of (34) follows from Proposition F.9. □

Proof of Lemma I.2. Fix a family $\mathcal{W}$ of compact and convex payoff sets and a state y. Let w be a corner of $\mathcal{B}_y(\mathcal{W})$ and suppose towards a contradiction that $(w, N) \notin \Gamma_y(\mathcal{W})$ for some non-extremal outward normal vector $N \notin \{\pm e_i\}$. Since $\Gamma_y(\mathcal{W})$ is closed, $(w', N') \notin \Gamma_y(\mathcal{W})$ for (w', N') sufficiently close to (w, N). A solution to (34) with initial conditions $(w - \varepsilon N, N)$ for $\varepsilon > 0$ sufficiently small thus cuts through $\mathcal{B}_y(\mathcal{W})$ and satisfies all properties of Lemma I.3—an impossibility. Because $\Gamma_y(\mathcal{W})$ is closed, we deduce that $(w, N) \in \Gamma_y(\mathcal{W})$ for all outward normal vectors N.

It follows that for any payoff-direction pair $(w, N_w) \in \mathcal{N}_{\mathcal{B}_y(\mathcal{W})} \setminus \Gamma_y(\mathcal{W})$, there exists a sufficiently small neighborhood U of w such that $\partial \mathcal{B}_y(\mathcal{W})$ is smooth in U with $\inf_{v \in \partial \mathcal{B}_y(\mathcal{W}) \cap U} N_v^\top N_w > 0$. Fix $w_0 \in \partial \mathcal{B}_y(\mathcal{W}) \cap U$ and a neighborhood $V \subseteq U$ of w_0 such that N_v is bounded away from a coordinate direction for any $v \in \partial \mathcal{B}_y(\mathcal{W}) \cap V$. Let π denote the projection of V onto $\partial \mathcal{B}_y(\mathcal{W})$ in the direction N_{w_0}, define Lipschitz expansion $\mathcal{L}$ as in Lemma I.3, and let $\mathcal{C}$ be a solution to (34). Because $\Gamma_y(\mathcal{W})$ is closed, there is a neighborhood of (w_0, N_{w_0}) that is not contained in $\Gamma_y(\mathcal{W})$ either, on which solutions to (34) are continuous in initial conditions by Proposition F.9. Suppose towards a contradiction that $\mathcal{C}$ escapes $\mathcal{B}_y(\mathcal{W})$ within V. Then the solution $\mathcal{C}'$ to (34) for a slightly rotated initial condition cuts through $\mathcal{B}_y(\mathcal{W})$ as in Figure 20, satisfying all the conditions of Lemma I.3—an impossibility. If $\mathcal{C}$ falls into the interior of $\mathcal{B}_y(\mathcal{W})$, then the solution $\mathcal{C}'$ to (34) for a slightly rotated initial condition leaves and enters $\mathcal{B}_y(\mathcal{W})$ as in the left panel of Figure 5. By Proposition F.9, the maxima in (34) are taken over non-empty sets at any $v \in \mathcal{C}'$. Lemma I.1 implies that the solution W to (5) with A, β, and δ given by the maximizers remains on $\mathcal{C}'$ until an end point of $\mathcal{C}'$ is reached or a state transition occurs. By Lemma H.2, $\mathcal{C}' \subseteq \mathcal{B}_y(\mathcal{W})$, a contradiction. Since $h(v) = 0$ on the boundary, this shows that the boundary solves (9) on V. $\square$

I.2 Incentives provided through state transitions

In this appendix we show that the boundary of $\mathcal{B}_y(\mathcal{W})$ is characterized by the state-transition optimality equation within $\Gamma_y(\mathcal{W})$. Moreover, we establish that enforceable solutions to SDE (5) that solve the state-transition optimality equation cannot escape $\mathcal{B}_y(\mathcal{W})$. By Lemma I.1, this can happen only where the direction of the drift changes. We begin with the following auxiliary lemma.

Lemma I.4. *Consider a payoff pair w, action profiles $\alpha_1, \dots, \alpha_m$, incentives $\delta_k \in \Psi_{y,\alpha_k}(w, \mathcal{W})$ for $k = 1, \dots, m$, and weights $\eta_1, \dots, \eta_m \geq 0$ that sum up to 1, and define*

$$u := \sum_{k=1}^{m} \eta_k \big(g(y, \alpha_k) + \delta_k \lambda(y, \alpha_k) \big). \tag{44}$$

There exists an enforceable solution W to (5) with $W_{\tau_1} \in \mathcal{W}_{S_{\tau_1}}$ such that:

(i) If $w = u$, then W remains at w until time τ_1. This implies $w \in \mathcal{B}_y(\mathcal{W})$.

(ii) If w lies on a straight line between u and some payoff pair v, then W remains at w until it either jumps to v or a state transition occurs. If $v \in \mathcal{B}_y(\mathcal{W})$, this implies $w \in \mathcal{B}_y(\mathcal{W})$.

Proof. Suppose that there exist such payoff pairs w and v, supports $\mathcal{A}_\diamond^{(1)}, \dots, \mathcal{A}_\diamond^{(m)}$, pairs (δ_k, α_k) in $\Upsilon_{\mathcal{A}_\diamond^{(k)}}$, and weights η_k for $k = 1, \dots, m$. Consider a solution $(W, S, A, \beta, \delta, M)$ to (5) on a stochastic framework $(\Omega, \mathcal{F}, \mathbb{F}, P, Z, (J^{y,y'})_{(y,y') \in \mathcal{Z}})$ with $W_0 = w$ such that:

(i) $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is rich enough to admit partitions $\Xi_t = (\Xi_t^k)_k$ of Ω with $P(\Xi_t^k) = \eta_k$ for each $k = 1, \dots, m$ and each $t > 0$ such that $\Xi = (\Xi_t)_{t \geq 0}$ is independent of Z,

$(J^{y,y'})_{(y,y')\in\mathcal{Z}}$ and Ξ satisfies an exact law of large numbers; see Sun (2006),

(ii) Before the first jump time τ_1 of any of the processes $(J^{y,y'})_{(y,y')\in\mathcal{Z}}$, we have

$$A = \sum_{k=1}^{m} \alpha_k 1_{\Xi^k}, \qquad \delta = \sum_{k=1}^{m} \delta_k 1_{\Xi^k},$$

$\beta \equiv 0$, and $\mathrm{d}M_t = (v - w)(\mathrm{d}J'_t - \lambda' \,\mathrm{d}t)$, where J' is a Poisson process independent of $(J^{y,y'})_{(y,y')\in\mathcal{Z}}$ with intensity $\lambda' = \|u - w\| / \|v - w\|$.

It follows from an exact law of large numbers (see Proposition 2.5 in Sun (2006)) that

$$\mathrm{d}W_t = \sum_{y\in Y} \delta(y)\,\mathrm{d}J^y_t + (v - w)\,\mathrm{d}J'_t.$$

Thus, W stays at w until either a jump in J' or a state transition occurs. Moreover, $(0, \delta)$ enforces A with $W + r\delta(y') \in \mathcal{W}_{y'}$ for each y' on $[\![0, \sigma)\!)$. Let τ denote the first jump time of J'. On the event $\{\tau < \tau_1\}$, the process W jumps from w to $v \in \mathcal{B}(\mathcal{W})$. Thus, Lemma H.2 shows $w \in \mathcal{B}(\mathcal{W})$. □

We have shown that the state-transition optimality equation (8) has a solution outside of $\operatorname{int} \bar{\mathcal{S}}_y(\mathcal{W})$. It is a direct consequence of Lemma I.4 that all payoffs in $\bar{\mathcal{S}}_y(\mathcal{W})$ are contained in $\mathcal{B}_y(\mathcal{W})$, hence payoffs where (8) has no solution are in the interior of $\mathcal{B}_y(\mathcal{W})$.

Proof of Lemma 7.4. Fix $w \in \bar{\mathcal{S}}_y(\mathcal{W})$. Since $w \in \mathcal{D}_y(w)$, it is a convex combination of w_k decomposed by (α_k, δ_k) with weights $\eta_k \geq 0$. By Lemma I.4, $w \in \mathcal{B}_y(\mathcal{W})$. □

The following lemma is the key ingredient to characterize the boundary within $\Gamma_y(\mathcal{W})$.

Lemma I.5. *For any $\mathcal{A}_\diamond \subseteq \mathcal{A}(y)$, any $w \in \partial\mathcal{B}_y(\mathcal{W}) \setminus \partial\mathcal{K}_{y,\mathcal{A}_\diamond}(\mathcal{W})$, and any outward normal vector N, there exists no $(\alpha, \delta) \in \Upsilon_{\mathcal{A}_\diamond}(w)$ with $N^\top\big(g(y, \alpha) + \delta\lambda(y, \alpha) - w\big) > 0$.*

Proof of Lemma I.5. Fix a state y, support $\mathcal{A}_\diamond \subseteq \mathcal{A}(y)$, a payoff pair $w \in \partial\mathcal{B}_y(\mathcal{W}) \setminus \partial\mathcal{K}_{y,\mathcal{A}_\diamond}(\mathcal{W})$, and an outward normal vector N to $\mathcal{B}_y(\mathcal{W})$ at w. Suppose towards a contradiction that there exists $(\alpha, \delta) \in \Upsilon_{\mathcal{A}_\diamond}(w)$, for which $N^\top\big(g(y, \alpha) + \delta\lambda(y, \alpha) - w\big) > 0$. In particular, such w must lie in the interior of $\mathcal{K}_{y,\mathcal{A}_\diamond}(\mathcal{W})$. Because $w \mapsto \Upsilon_{\mathcal{A}_\diamond}(w)$ is continuous by Lemma F.8, for any w' sufficiently close to w there exist $(\alpha', \delta') \in \Upsilon_{\mathcal{A}_\diamond}(w')$ sufficiently close to (α, δ) such that $N^\top\big(g(y, \alpha') + \delta'\lambda(y, \alpha') - w'\big) > 0$. In particular, there exists $w' \notin \operatorname{cl} \mathcal{B}_y(\mathcal{W})$ close enough to w such that the straight line segment through $u := g(y, \alpha') + \delta'\lambda(y, \alpha')$ and

w' contains a point $v \in \operatorname{int} \mathcal{B}(\mathcal{W})$; see the left panel of Figure 5. Lemma I.4 implies that $w' \in \mathcal{B}_y(\mathcal{W})$, a contradiction. $\square$

Lemma I.6. *Fix any $w \in \partial\mathcal{B}_y(\mathcal{W})$ with $(w, N) \in \Gamma_y(\mathcal{W})$ for some outward normal N. Then $\partial\mathcal{B}_y(\mathcal{W})$ is a continuously differentiable solution to* (8) *outside of $\mathcal{S}_y(\mathcal{W}) \cup \mathcal{K}_y(\mathcal{W})$.*

Proof. Fix any $w \in \partial\mathcal{B}_y(\mathcal{W}) \setminus \mathcal{K}_y(\mathcal{W})$ with outward normal vector N such that $(w, N) \in \Gamma_y(\mathcal{W})$. By definition of $\Gamma_y(\mathcal{W})$, there exists $\mathcal{A}_\diamond$ and $(\alpha, \delta) \in \Upsilon_{\mathcal{A}_\diamond}(w)$ with $N^\top\big(g(y, a) + \delta\lambda(y, a) - w\big) \geq 0$. By Lemma I.5, the inequality cannot be strict at $w \notin \mathcal{K}_y(\mathcal{W})$, hence (w, N) solves (8).

Suppose now that there are multiple outward normal vectors to $\mathcal{B}_y(\mathcal{W})$ at w. Then Lemma I.2 implies that $(w, N) \in \Gamma_y(\mathcal{W})$ for any outward normal N. Fix three distinct outward normal vectors N_1, N_2, N_3. Because $\mathcal{A}(y)$ is finite, we can choose these vectors such that $(w, N_k) \in \Gamma_{y,\mathcal{A}_\diamond}(\mathcal{W})$ for the same set $\mathcal{A}_\diamond \subseteq \mathcal{A}(y)$. It follows from Lemma I.5 that for each k,

$$N_k{}^\top w = \max_{x \,\in\, \operatorname{conv} \mathcal{D}_{\mathcal{A}_\diamond}(w)} N_k{}^\top x_k = \max_{x \in \mathcal{D}_{\mathcal{A}_\diamond}(w)} N_k{}^\top x_k.$$

However, w can be tangential to $\operatorname{conv} \mathcal{D}_{\mathcal{A}_\diamond}(w)$ in three distinct directions only if w is an extreme point of $\operatorname{conv} \mathcal{D}_{\mathcal{A}_\diamond}(w)$, hence $w \in \mathcal{D}_{\mathcal{A}_\diamond}(w)$. By Lemma 6.2 this implies $w \in \mathcal{S}_y(\mathcal{W})$. $\square$

It remains to show that the strategy profile that remains on a solution to (8) does not escape $\mathcal{B}_y(\mathcal{W})$ at corners.

Lemma I.7. *Consider a compact and convex set $\mathcal{X}$, whose boundary is a solution to* (9) *outside of $\Gamma_y(\mathcal{W})$. Any non-stationary corner w of $\mathcal{X}$ satisfies one of the following conditions:*

(i) w is attainable by a solution to SDE (5) *described in Lemma I.4 for some $v \in \partial\mathcal{X}$.*

(ii) w is a starting point of a solution $\mathcal{C}$ to (8)*, i.e., the solution W to SDE* (5) *for maximizers A and δ of the state-transition optimality equation* (8) *locally stays on $\mathcal{C}$.*

Proof. Fix a non-stationary corner w of $\mathcal{X}$. Lemma I.2 implies $(w, N) \in \Gamma_y(\mathcal{W})$ for any outward normal vector N, which is equivalent to $\operatorname{conv} \mathcal{D}_y(w) \cap C_w \neq \emptyset$, where C_w is the normal cone to $\mathcal{X}$ at w. If $w \in \operatorname{conv} \mathcal{D}_y(w)$, then condition (i) of Lemma I.4 is satisfied, hence $w \in \mathcal{B}_y(\mathcal{W})$. Suppose, therefore, that $\operatorname{conv} \mathcal{D}_y(w)$ is strictly separated from w. If there exists $u \in \operatorname{conv} \mathcal{D}_y(w) \cap \operatorname{int} C_w$, then the straight line L through w and u intersects the interior of $\mathcal{X}$. Let $v \neq w$ denote the other intersection point of L with $\partial\mathcal{X}$. Then w and v

satisfy condition (ii) of Lemma I.4. Finally, suppose that $\operatorname{conv} \mathcal{D}_y(w)$ intersects only ∂C_w. Then $\mathcal{D}_y(w)$ must intersect ∂C_w, hence there exists an extremal normal vector such that (8) holds. Moreover, w must be a starting point of a solution to (8) because $\max_{v \in \mathcal{D}_y(w)} N^\top v \geq 0$ for all outward normal vectors N. $\square$

I.3 Proof of Theorem 7.5

Proof of Theorem 7.5. Fix a payoff set $\mathcal{X}$ that satisfies the conditions of Theorem 7.5 in state y. We will show that $\partial \mathcal{X}$ is generated by $\mathcal{W}$. Consider first a boundary payoff w, where (w, N_w) solves (8) and the direction of the drift changes. Then w satisfies Condition (i) of Lemma I.4, hence $w \in \mathcal{B}_y(\mathcal{W})$. Any stationary corner of $\mathcal{X}$ lies in $\mathcal{B}_y(\mathcal{W})$ by Lemma 7.4. Lemma I.7 implies that any other corner is either a starting point of (8) or it satisfies one of the the conditions of Lemma I.4 for some $v \in \partial \mathcal{X}$. If w satisfies Condition (i), then $w \in \mathcal{B}_y(\mathcal{W})$ by Lemma I.4. If Condition (ii) is satisfied instead, then an enforceable solution to (5) locally remains on the boundary of $\partial \mathcal{X}$. Together with Lemma I.1 it follows that every boundary payoff can be attained locally by a restricted-enforceable solution to (5) that remains on the boundary, hence $\operatorname{cl} \mathcal{X} \subseteq \mathcal{B}_y(\mathcal{W})$.

Next, Lemmas I.2, I.6, and I.7 show that the boundary of $\mathcal{B}_y(\mathcal{W})$ satisfies the conditions of Theorem 7.5. This implies that $\operatorname{cl} \mathcal{B}_y(\mathcal{W}) \subseteq \mathcal{B}_y(\mathcal{W})$, hence $\mathcal{B}_y(\mathcal{W})$ is closed. Finally, $\mathcal{B}_y(\mathcal{W}) \subseteq \mathcal{V}^*_y(\mathcal{W})$ by Corollary D.2 and $\mathcal{S}_y(\mathcal{W}) \subseteq \mathcal{B}_y(\mathcal{W})$ by Lemma 7.4. $\square$

References

Bernard, Benjamin. 2024. "Continuous-Time Games with Imperfect and Abrupt Information." *The Review of Economic Studies*, **91(4)**: 1988–2052.

Jacod, J., and A. Shiryaev. 2002. *Limit Theorems for Stochastic Processes. Grundlehren der mathematischen Wissenschaften*. 2 ed., Springer Berlin Heidelberg.

Kazamaki, N. 2006. *Continuous Exponential Martingales and BMO. Lecture Notes in Mathematics*, Springer Berlin Heidelberg.

Protter, P.E. 2005. *Stochastic Integration and Differential Equations. Stochastic Modelling and Applied Probability*. 2 ed., Springer Berlin Heidelberg.

Sun, Yeneng. 2006. "The exact law of large numbers via Fubini extension and characterization of insurable risks." *Journal of Economic Theory*, **126(1)**: 31–69.