Online Appendix to "Continuous-Time Stochastic Games with Imperfect Public Monitoring"

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Appendices D and E contain supplementary information. Appendix F establishes regularity of the optimality equations in Propositions 7.3 and F.9. Appendix G contains the proofs that characterize $\mathcal{S}_y(\mathcal{W})$ and $\mathcal{K}_y(\mathcal{W})$ under our various assumptions. Proofs that only require minor adaptations from Bernard (2024) are deferred to Appendices H and I in the supplemental information file found online.[13]

D Computation

D.1 Individually rational payoffs

Theorem 7.5 characterizes $\mathcal{B}_y(\mathcal{W})$ as the largest $\mathcal{W}$-feasible set that satisfies certain properties. We can sharpen the upper bound by additionally imposing individual rationality.

Definition D.1. Fix a family $\mathcal{W} = (\mathcal{W}_y)_{y\in\mathcal{Y}}$ of payoff sets.

(i) Let $\underline{w}^i_{y'}$ denote player i's lowest payoff in $\mathcal{W}_{y'}$. Player i's $\mathcal{W}$-*minmax payoff* in state y is

$$\underline{v}^i_y(\mathcal{W}) := \min_{\alpha^{-i}\in\Delta\mathcal{A}^{-i}(y)} \max_{a^i\in\mathcal{A}^i(y)} g^i(y, a^i, \alpha^{-i}) + \frac{1}{r}\sum_{y'\in\mathcal{Y}} (\underline{w}^i_{y'} - \underline{v}^i_y(\mathcal{W}))\lambda_{y,y'}(a^i, \alpha^{-i}). \quad (28)$$

(ii) Any payoff pair w with $w^i \geq \underline{v}^i_y(\mathcal{W})$ for $i = 1, 2$ is $\mathcal{W}$-*individually rational* in state y.

We denote by $\mathcal{V}^*_y(r;\mathcal{W})$ or $\mathcal{V}^*_y(\mathcal{W})$ the set of all $\mathcal{W}$-individually rational payoffs in $\mathcal{V}_y(r;\mathcal{W})$. We denote by $\underline{v}^i_{p,y}(\mathcal{W})$ the pure-action $\mathcal{W}$-*minmax payoff* in state y and by $\mathcal{V}^{p*}_y(\mathcal{W})$ the set of all pure-action $\mathcal{W}$-individually rational payoffs in $\mathcal{V}_y(r;\mathcal{W})$.

Any restricted-enforceable solution to (5) must be $\mathcal{W}$-individually rational as, otherwise, one player can profitably deviate to the strategy of myopic best responses. This gives rise to the following corollary to Lemma 5.2.

Corollary D.2. *$\mathcal{B}_y(\mathcal{W}) \subseteq \mathcal{V}^*_y(\mathcal{W})$ and $\mathcal{B}^p_y(\mathcal{W}) \subseteq \mathcal{V}^{p*}_y(\mathcal{W})$.*

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[13]See here: https://benjamin-bernard.com/research/stochastic-games_SI.pdf.

In a related discrete-time model where the public signal becomes perfectly informative as the length of the time period shrinks to 0, Pęski and Wiseman (2015) show that $\mathcal{W}$-feasibility and $\mathcal{W}$-individual rationality are the only constraints imposed on $\mathcal{B}_y(\mathcal{W})$ in the limit. Our Theorem 7.5 shows that this is not true away from the limit.

D.2 Starting with a tighter bound

For any non-singleton communicating class, the computation time through Proposition 4.9 can be reduced by starting with a tighter upper bound. Call a family of payoff sets $\mathcal{W}$ *self-feasible* and *self-individually rational* if it is $\mathcal{W}$-feasible and $\mathcal{W}$-individually rational, respectively.

(i) Call the largest family of self-feasible payoffs the *feasible payoffs* $\mathcal{V}(r) = (\mathcal{V}_y(r))_{y\in\mathcal{Y}}$.

(ii) Call the largest family of self-feasible and self-individually rational payoffs the family $\mathcal{V}^*(r) = (\mathcal{V}^*_y(r))_{y\in\mathcal{Y}}$ of *feasible and individually rational payoffs.*

Since $\mathcal{E}(r)$ is self-feasible and self-individually rational, it follows that $\mathcal{E}(r;y) \subseteq \mathcal{V}^*_y(r)$. Consequently, the algorithms in Propositions 6.8 and 4.9 can be started with $\mathcal{V}^*(r)$.

There are various equivalent characterizations of $\mathcal{V}^*(r)$. If the number of pure actions in each state is low, it is convenient to compute $\mathcal{V}_y(r)$ as the convex hull of all stationary pure-strategy payoffs; see Blackwell (1965). Then $\mathcal{V}^*_y(r)$ is the set of all $w \in \mathcal{V}_y(r)$ that satisfy $w^i \geq \underline{v}^i_y(r)$ for $i = 1, 2$, where the family $(\underline{v}^i_y(r))_{y\in\mathcal{Y}}$ of minmax payoffs solves the system

$$\underline{v}^i_y(r) = \min_{a_{-i}\in\mathcal{A}_{-i}(y)} \max_{\alpha_i\in\Delta(\mathcal{A}_i(y))} g(y,a) + \sum_{y'\in\mathcal{Y}} \frac{\underline{v}^i_{y'}(r) - \underline{v}^i_y(r)}{r} \lambda_{y,y'}(a^i, \alpha^{-i}).$$

If the number of pure actions is large, $\mathcal{V}^*(r)$ can be computed by iteratively applying the operator $\mathcal{V}^*$ defined in Appendix D.1 to $(\mathcal{V}^*_0, \ldots, \mathcal{V}^*_0)$ until it converges to $\mathcal{V}^*(r)$.

D.3 Implementing the local inclusion in Section 6.2

The first step to computing $\mathcal{S}_{y,\mathcal{A}_\diamond}(\mathcal{W})$ through Lemma 6.3 is to find an extremal stationary payoff w_0, from which we can solve the local inclusion. We can do this in one of two ways:

(i) If we are interested in $\mathcal{S}_{y,\mathcal{A}_\diamond}(\mathcal{W})$ for the sake of characterizing $\mathcal{B}_y(\mathcal{W})$, then we simply solve the optimality equations until we find a stationary payoff pair, and such a stationary payoff pair must be extremal. If we find none, there are no extremal stationary payoffs.

(ii) In general, we start by finding *some* stationary payoff. We can either compute $\mathcal{D}_{\mathcal{A}_\diamond}(w_n)$

for some grid of payoff pairs $(w_n)_{n\geq 0}$ and see if one of them is stationary,[14] or we can compute $\mathcal{S}_{y,\mathcal{A}_\diamond}(\alpha_n, \mathcal{W})$ defined in Footnote 7 for a grid $(\alpha_n)_{n\geq 0}$ of mixed action profiles supported on $\mathcal{A}_\diamond$. Once we find some stationary payoff $\hat{w}$, Lemma 6.2 implies $\hat{w} + tN$ for an arbitrary direction N will lie in $\partial\mathcal{D}_{\mathcal{A}_\diamond}(\hat{w} + tN)$ for some t sufficiently large.

From an extremal stationary payoff w_0, we can solve the local inclusion as follows. Let θ_0 denote an angle such that $w_0 + \varepsilon N(\theta_0) \notin \mathcal{S}_{y,\mathcal{A}_\diamond}(\mathcal{W})$ for any $\varepsilon > 0$ and fix a step size $\Delta w > 0$. Let $T(\theta)$ denote the counterclockwise 90° rotation of $N(\theta)$. Let θ_{k+1} denote the smallest angle $\theta \geq \theta_k - \pi/2$ such that $w_k + \Delta w T(\theta) \in \partial\mathcal{D}_{\mathcal{A}_\diamond}(w_k + \Delta w T(\theta))$ and set $w_{k+1} = w_k + \Delta w T(\theta)$. Since $\mathcal{S}_{y,\mathcal{A}_\diamond}(\mathcal{W})$ may not be convex, θ_{k+1} may be smaller than θ_k, but it has to be at least $\theta_k - \pi/2$ since w_k is known to be extremal in direction $N(\theta_k)$. If the conditions of Lemma 6.4 are satisfied and $\mathcal{S}_{y,\mathcal{A}_\diamond}(\mathcal{W})$ is known to be simply connected, the found solution is the entire boundary. If $\mathcal{S}_{y,\mathcal{A}_\diamond}(\mathcal{W})$ is not known to be simply connected, we repeat the entire process by trying to find stationary payoffs in other connected components with approach (i) or (ii).

E Boundary segments in $\mathcal{K}_y^p(\mathcal{W})$ of positive length

This section provides an example where $\partial\mathcal{E}^p(y)$ contains segments of positive length in $\mathcal{K}_y(\mathcal{E}^p)$, along which the drift of any PPE points towards the interior of $\mathcal{E}^p(y)$. Moreover, the example shows that the PPE payoff correspondence is not monotone in either λ or r if the other is held fixed. Consider a two-state absorbing game with initial state y_1, absorbing state y_2, discount rate $r = 1$, $\mu(y, a) \equiv 0$, as well as flow payoffs and intensity of state transitions given in Figure 14. Since the public signal is completely uninformative, enforceable action profiles must maximize each player's flow payoff over all unilateral deviations that leave the transition intensity unchanged. As a result, $a_A = (A, A)$ is the only enforceable action profile a with $\lambda_{y_1,y_2}(a) = 0$ and $a_C = (C, C)$ is the only enforceable action profile a with $\lambda_{y_1,y_2}(a) = 2$. It is easy to check that $a_B = (B, B)$ is enforced by $\delta \in [-1, 1]^2$. In particular, only the static Nash profiles are enforceable in either state; see Figure 14.

The PPE payoff correspondence is shown in Figure 15. Since y_2 is absorbing and players receive no information, $\mathcal{E}(y_2)$ is simply the convex hull of static Nash payoffs. In particular, player 2's continuation value after a state transition to y_2 is at least $\underline{w}_{y_2}^2 = 3$. Because player 2 also earns an expected flow payoff of 3 in state y_1 if a_A or a_C are played, it follows that a_B must

[14]Because of statement (iii) of Lemma 6.2, if $w_k \notin \mathcal{D}_{\mathcal{A}_\diamond}(w_k)$, it is most efficient to choose w_{k+1} in the direction of $\mathcal{D}_{\mathcal{A}_\diamond}(w_k)$. If $\mathcal{D}_{\mathcal{A}_\diamond}(w_k)$ becomes empty before w_k reaches $\mathcal{D}_{\mathcal{A}_\diamond}(w_k)$, then $\mathcal{S}_{y,\mathcal{A}_\diamond}(\mathcal{W})$ may be empty.

	A	B	C
A	1, 3	3, 0	0, −1
B	0, −1	4, 0	0, −1
C	0, −1	3, 0	7, 3

$g(y_1, a)$

	A	B	C
A	0	0	0
B	0	1	2
C	0	2	2

$\lambda(y_1, a)$

	A	B	C	D
A	0, 4	1, 3	7, 3	8, 4
B	0, 2	1, 3	7, 3	8, 2

$g(y_2, a)$

Figure 14: The Nash equilibria (circled) are the only enforceable action profiles.

be played in any PPE that delivers a payoff below 3 to player 2. Since $\delta^2 \leq 1$ is required to enforce a_B, player 2 must receive at least 2 in any PPE—otherwise no promise from $\mathcal{E}(y_2)$ would enforce a_B. Because $g^2(y_1, a_B) + \delta^2 \lambda(y_1, a_B) \leq 1$ for any such δ, it follows that the drift points strictly towards the interior of $\mathcal{E}(y_1)$ at the lower boundary of $\mathcal{E}(y_1)$. Even though there is slack left from the inward-drift condition, a_B cannot be credibly enforced outside of $\mathcal{K}_{y_1, a_B}(\mathcal{E})$ because continuation values from $\mathcal{E}(y_2)$ are too rewarding to play of a_B. If the reward exceeds 1, player 2 prefers to deviate to C to increase the frequency of state transitions.

Payoff pair w_0 on the lower boundary can be attained, for example, by a restricted-enforceable solution W to (5) with $A = a_B$, $\delta = \delta_{w_0}(y_2) := (w_0 - g(y_1, a_B))/2$, and $\beta = M = 0$ until $\operatorname{conv} \mathcal{S}(\mathcal{E})$ is reached; see Figure 15. In this example, the straight line segments between w_L and $g(y_1, a_A)$ and between w_R and $g(y_1, a_C)$ are straight solutions to (8).

Player 2's payoff is bounded away from their minmax payoff $\underline{v}^2_{p, y_1} = 5/3$, attained at minmax profile (B, C), found by solving (28) for $\underline{w}^2_{y_2} = 3$. We observe that $\mathcal{E}(y_1)$ is bounded away from the set of feasible and individually rational payoffs for a different reason than in Figure 1—not because value has to be burnt to incentivize a_B, and not because players are impatient. The reason is specific to stochastic games and arises when there is a large payoff difference between states: a larger difference between $\mathcal{V}_y(\mathcal{W})$ and continuation values in $\mathcal{W}_{y'}$ implies that state transitions have a larger impact on incentives relative to the flow payoffs. Consequently, the set of enforceable action profiles in state y shrinks with this difference and action profiles that attain extremal feasible payoffs may not be enforceable.

This issue is not mitigated but amplified by greater patience because a larger weight is attached to the continuation value after a state transition. Here, $\mathcal{K}_{y_1, a_B}(\mathcal{E}(r))$ shrinks linearly with r, hence there are extremal PPE payoffs with strict inward drift for any $r \in (0, 1]$.

This game shows that the PPE payoff correspondence is not monotone in r or λ individually. Note that $\mathcal{E}(y_2)$ is independent of r and $\mathcal{E}(y_1)$ converges to $\mathcal{E}(y_2)$ as the entire payoff weight

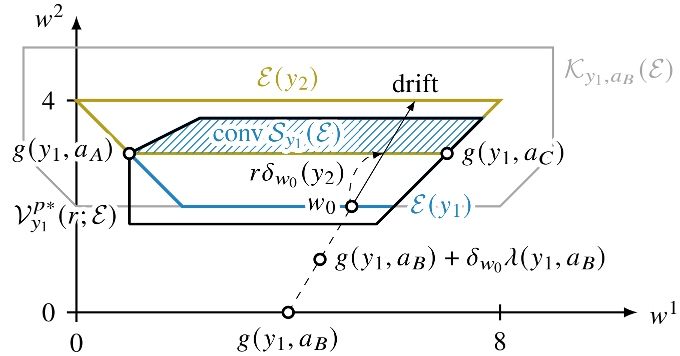

Figure 15: An absorbing game with strictly inward-pointing drift at $\partial\mathcal{E}(y_1)$.

is shifted to y_2 in the limit as $r \to 0$.[15] While the upper frontier of $\mathcal{E}(y_1)$ expands as $r \to 0$, the lower frontier contracts. Similarly, an increase in the frequency of state transitions by a factor $\eta > 1$ attaches a larger weight to the absorbing state y_2. Expected continuation promises remain the same if the delivered promises δ are divided by η. Correspondingly, $\mathcal{K}_{y_1,a_B}(\mathcal{E})$ shrinks proportionally to $1/\eta$, i.e., the lower bound of $\mathcal{E}(y_1)$ rises with η.

F Regularity of the optimality equations

To show local Lipschitz continuity of the optimality equations, we show that both are a maximization of a locally Lipschitz continuous function over a locally Lipschitz continuous set of actions and incentives. For set-valued maps, Lipschitz continuity is defined as follows.

Definition F.1. A set-valued map $G : x \mapsto G(x) \subseteq \mathbb{R}^n$ is said to be *Lipschitz continuous* if there exists a constant K such that $G(x) \subseteq G(\tilde{x}) + K\|x - \tilde{x}\|B_1(0)$ for any x and $\tilde{x}$, where $B_1(0)$ denotes the closed unit ball in $\mathbb{R}^n$ centered at the origin and + is the setwise addition.

Lemma F.2. *Let $f(x, y)$ be a function and let $G(x)$ be a set-valued map, both (locally) Lipschitz-continuous. Then $h(x) = \max_{y\in G(x)} f(x, y)$ is (locally) Lipschitz continuous.*

Because the enforceability constraints define a closed convex polyhedron by Lemma 5.5 and $\mathcal{W}$-feasibility is a closed convex set, we begin with four preliminary lemmas about continuity properties of the intersection of such sets.

Lemma F.3. *Let F and G be closed- and convex-valued maps that are locally Lipschitz continuous at $x_0 \in \text{dom}\,F \cap \text{dom}\,G$ such that one of them is locally bounded at x_0. If $F(x_0) \cap \text{int}\,G(x_0) \neq \emptyset$, then $F \cap G$ is locally Lipschitz continuous at x_0.*

[15] Relevant for the convergence is that at all extremal payoff pairs of except $g(y_1, a_A)$, action profiles a_B or a_C are played, for which state transitions happen with positive frequency.

The proofs of Lemmas F.2 and F.3 have appeared in Bernard (2024).

Lemma F.4. *Fix a matrix $A \in \mathbb{R}^{m\times n}$, a concave function $f : \mathbb{R}^d \to \mathbb{R}^m$, and a compact and convex set $G \subseteq \mathbb{R}^n$. For any $x \in \mathbb{R}^d$, let $F(x) \subseteq \mathbb{R}^n$ be the set of all $z \in \mathbb{R}^n$ that satisfy*

$$Az \le f(x). \tag{29}$$

Then $F \cap G$ is locally Lipschitz continuous on the interior of $\mathcal{D} = \{x \mid F(x) \cap G \ne \emptyset\}$.

Proof. We begin with the following two observations.

Claim 1. For any $x_0, x_1 \in \mathcal{D}$, any $z_k \in F(x_k) \cap G$ for $k = 0, 1$, and any $\eta \in (0, 1)$, define

$$x_\eta := \eta x_1 + (1-\eta)x_0, \qquad\qquad z_\eta := \eta z_1 + (1-\eta)z_0. \tag{30}$$

Then $z_\eta \in F(x_\eta) \cap G$, hence the domain $\mathcal{D}$ and the image $\mathcal{I} = \bigcup_{x\in\mathcal{D}} F(x) \cap G$ are both convex.

To show convexity of $\mathcal{D}$, start with $x_0, x_1 \in \mathcal{D}$ and $\eta \in (0, 1)$ and define x_η and z_η according to (30) for arbitrary $z_k \in F(x_k) \cap G$. For convexity of $\mathcal{I}$, start with $z_0, z_1 \in \bigcup_{x\in\mathcal{D}} F(x) \cap G$ and define z_η and x_η according to (30) for arbitrary x_k with $z_k \in F(x_k) \cap G$. In either case,

$$Az_\eta = \eta Az_1 + (1-\eta)Az_0 \le \eta f(x_0) + (1-\eta) f(x_1) \le f(x_\eta),$$

by concavity of f, hence $z_\eta \in F(x_\eta)$. Since G is convex, it follows that $z_\eta \in G$, hence $x_\eta \in \mathcal{D}$.

Claim 2. If there exists $x \in \mathcal{D}$ with $F(x) \cap \operatorname{int} G \ne \emptyset$, then $F \cap G$ is locally Lipschitz on $\operatorname{int} \mathcal{D}$.

Observe first that f is locally Lipschitz continuous because it is concave, hence F is locally Lipschitz continuous; see Hoffman (1952). Fix such x_0 and $z_0 \in F(x_0) \cap \operatorname{int} G$ and an arbitrary point $x \in \operatorname{int} \mathcal{D}$. Since $\mathcal{D}$ is convex by Claim 1, we can write $x = \eta x_1 + (1-\eta)x_0$ for some other $x_1 \in \mathcal{D}$ and $\eta \in (0, 1)$. For any $z_1 \in F(x_1) \cap G$, the vector z_η defined as above lies in $z_\eta \in F(x) \cap \operatorname{int} G$. Thus, Lemma F.3 shows that $F \cap G$ is locally Lipschitz continuous at x.

The proof is concluded by rewriting F and G such that Claim 2 applies. Let H be the (possibly empty) intersection of all hyperplanes that contain $\mathcal{I}$. If $H = \mathbb{R}^n$, set $G' = G$. If H is lower-dimensional, let G' denote the pre-image of all vectors z whose orthogonal projection onto H lies in $\mathcal{I}$. In either case, $G = H \cap G'$ where G' is full-dimensional and convex since $\mathcal{I}$ is convex by Claim 1. Thus, there exist $x_0 \in \mathcal{D}$ and $z_0 \in F(x_0)$ such that z_0 lies in the relative interior of $\mathcal{I}$, i.e., $z_0 \in \operatorname{int} G'$. Because $F(x) \cap H$ is characterized by (29) with concave f, Claim 2 shows that $F(x) \cap H \cap G' = F(x) \cap G$ is locally Lipschitz continuous on $\operatorname{int} \mathcal{D}$. □

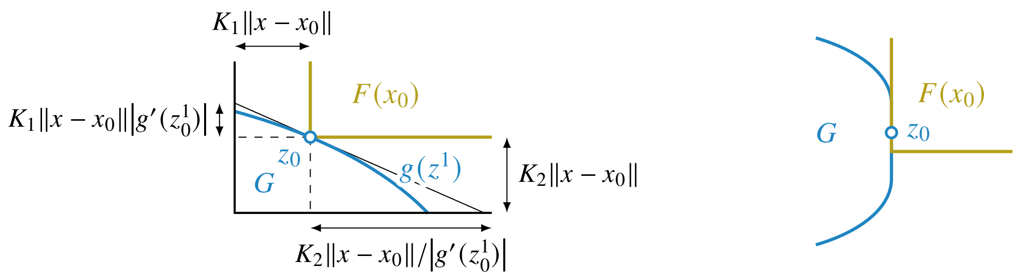

Figure 16: Illustration that $F(x_0) \cap G \subseteq \mathbb{R}^2$ is lower hemicontinuous.

Lemma F.5. *Let $G \subseteq \mathbb{R}^2$ be a compact and convex set. Let $F(x) \subseteq \mathbb{R}^2$ be the (possibly unbounded) rectangle of all $z \in \mathbb{R}^2$ that satisfy a subset of the constraints $f^i_\ell(x) \leq z^i$ and $z^i \leq f^i_h(x)$ for $i = 1, 2$, where each $f^i_\ell, f^i_h : \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz continuous. Then:*

(i) $F \cap G$ is continuous on its effective domain $\mathcal{D} = \{x \mid F(x) \cap G \neq \emptyset\}$.

(ii) If f^i_k is either constant or strictly monotone in any x^j for any $j = 1, \ldots, d$, any $i = 1, 2$, and any $k \in \{\ell, h\}$, then $F \cap G$ is locally Lipschitz continuous in the interior of $\mathcal{D}$.

The requirement that G is two-dimensional is crucial, since the analogue of neither statement is true in three dimensions without additional impositions on the convex set G.

Proof. $F(x)$ is locally Lipschitz continuous by Hoffman (1952). The closed graph theorem implies that $F \cap G$ is upper hemicontinuous. It follows from Lemma F.3 that $F \cap G$ is locally Lipschitz continuous at x if $F(x) \cap \operatorname{int} G \neq \emptyset$. It remains to show the continuity properties of $F \cap G$ at $x_0 \in \mathcal{D}$ where $F(x_0)$ intersects G only at the boundary. Since $F(x_0)$ is a rectangle, the intersection with G is either a single point or a segment parallel to a coordinate axis.

Suppose first that the intersection is a single point $\{z_0\}$ such that no coordinate direction is an outward normal to G at z_0. Then z_0 is a corner of $F(x_0)$ and, without loss of generality, suppose that $z_0 = f_\ell(x_0)$ as in the left panel of Figure 16. Moreover, the parametrization g of ∂G in a neighborhood of z_0 with respect to coordinate 2 has non-zero and finite derivative $g'(z_0^1)$. Fix a small neighborhood U of x_0 and let K_i be the local Lipschitz constants of f^i_ℓ on U. For any $x \in U$, the intersection $F(x) \cap G$ must be contained in the right triangle indicated in the left panel of Figure 16. Its legs are of length

$$L_1(x) = K_1\|x - x_0\| + K_2\|x - x_0\|\frac{1}{\left|g'(z_0^1)\right|}, \quad L_2(x) = K_2\|x - x_0\| + K_1\|x - x_0\|\left|g'(z_0^1)\right|.$$

Thus, $F \cap G$ is locally Lipschitz continuous, hence also lower hemicontinuous. for $x \in U$

Suppose next that the intersection $F(x_0) \cap G$ maximizes or minimizes G in a coordinate direction. Without loss of generality, suppose that all $z \in F(x_0) \cap G$ satisfy $z^1 = f_\ell^1(x_0)$ and G contains no vector z with $z^1 > f_\ell^1(x_0)$ as in the right panel of Figure 16. We first show lower hemicontinuity. Consider z_0 in the relative interior of $F(x_0) \cap G$, which must satisfy $z_0^2 \in (f_\ell^2(x_0), f_h^2(x_0))$. Since $F(x_0)$ intersects G only if $f_\ell^1(x) \leq f_\ell^1(x_0)$, it follows that $z_0 \in F(x) \cap G$ for any x sufficiently close to x_0 by continuity of f_ℓ^2 and f_h^2. Suppose next that z_0 maximizes or minimizes z^2 among $z \in F(x_0) \cap G$. Since G is convex, local parametrizations of the upper and lower boundary are continuous. Thus, z_0 is approximated by any $z_n \in F(x_n) \cap G$ on the upper or lower frontier of $F(x_n) \cap G$, respectively, for any sequence $(x_n)_{n \geq 1} \subseteq \mathcal{D}$ converging to x_0. Finally, we show local Lipschitz continuity at such $x_0 \in \operatorname{int} \mathcal{D}$ when each f_ℓ^i and f_h^i is either constant or strictly monotone. Consider still the setting in the right panel of Figure 16 where $f_\ell^1(x_0)$ binds. If f_ℓ^1 is strictly monotone in some x^j, then here exists x arbitrarily close to x_0 such that $F(x)$ does not intersect G, contradicting $x_0 \in \operatorname{int} \mathcal{D}$. Thus, $f_\ell^1(x)$ must be constant in x. Then $F \cap G$ is entirely contained in the vertical line segment $\left\{z \in \mathbb{R}^2 \;\middle|\; z^1 = f_\ell^1(x)\right\}$, hence the intersection changes locally Lipschitz continuously in f_ℓ^2 and f_h^2. $\square$

Lemma F.6. *Let $G_1, \dots, G_n \subseteq \mathbb{R}^2$ be compact and convex sets and set $G = G_1 \times \dots \times G_n$. Fix a matrix $A \in \mathbb{R}^{m \times 2n}$ such that each row A^k has non-zero entries only in either odd- or even-numbered columns, and an affine function $f : \mathbb{R}^d \to \mathbb{R}^m$. For any $x \in \mathbb{R}^d$, let $F(x) \subseteq \mathbb{R}^{2n}$ be the closed convex polyhedron of all $z \in \mathbb{R}^{2n}$ that satisfy $Az \leq f(x)$. Then $F \cap G$ is continuous on $\mathcal{D} = \{x \mid F(x) \cap G \neq \emptyset\}$ and locally Lipschitz continuous on* $\operatorname{int} \mathcal{D}$.

Proof. Local Lipschitz continuity follows straight from Lemma F.4. Since $F \cap G$ is upper hemicontinuous on $\mathcal{D}$ by the closed graph theorem, it remains to show lower hemicontinuity of $F \cap G$ at any $x_0 \in \partial \mathcal{D}$. Note again we may assume without loss of generality that each G_k has non-empty interior as we could otherwise incorporate their constraints directly into F. Let $\pi_k(z)$ denote the projection of z onto the components corresponding to set G_k.

Claim 3. For any $x_0 \in \partial \mathcal{D}$, there exists a non-empty set $K \subseteq \{1, \dots, n\}$ of indices such that $\pi_k(F(x_0) \cap G) \subseteq \partial G_k$ for all $k \in K$.

Fix $x_0 \in \mathcal{D}$ such that for each k, there exists $z_k \in F(x_0) \cap G$ with $\pi_k(z_k) \in \operatorname{int} G_k$. Let z_* denote any strict convex combination of all $z_1, \dots, z_n$, which lies in $F(x_0) \cap G$ by convexity. Moreover, since $\pi_k(z_j) \in G_k$ for each j, we must have $\pi_k(z_*) \in \operatorname{int} G_k$ for all k. This implies $z_* \in F(x_0) \cap \operatorname{int} G$, hence $F(x) \cap G \neq \emptyset$ for all x sufficiently close to x_0, hence $x_0 \operatorname{int} \mathcal{D}$.

Fix arbitrary $x_0 \in \operatorname{ext}\mathcal{D}$, approximated by a sequence $(x_n)_{n\geq 1} \subseteq \operatorname{ext}\mathcal{D}$. By Claim 3, for each n there exists a set of indices K^n such that $\pi_k(F(x_n) \cap G) \subseteq \partial G_k$ for all $k \in K^n$. Let us further decompose K^n into the disjoint union of indices K_s^n and K_ℓ^n, where K_s^n are those indices $k \in K^n$ where $\pi_k(F(x_0) \cap G)$ is a singleton $\{z_k^*\}$ and K_ℓ^n is the set of those indices $k \in K^n$, where $\pi_k(F(x_0) \cap G)$ is a line segment L_k of positive length. By passing to a subsequence, we may assume that those are the same set of indices $K_s^n = K_s$ and $K_\ell^n = K_\ell$ for each n.

The basic idea behind the proof is that the conditions on A imply that $\pi_k(F(x_0))$ is a rectangle that moves around G_k. For each $k \in K = K_s \cup K_\ell$, we use a construction as in Lemma F.5, and indices $k \notin K$ are not binding. We formalize this as follows. For $k \in K_\ell$, let P_k denote a closed inner polygon approximation of G_k that contains L_k and set

$$P_L = \bigtimes_{k\in K_\ell} P_k \times \bigtimes_{k\notin K_\ell} \mathbb{R}^2, \qquad G_S = \bigtimes_{k\in K_s} G_k \times \bigtimes_{k\notin K_s} \mathbb{R}^2, \qquad G_I = \bigtimes_{k\notin K} G_k \times \bigtimes_{k\in K} \mathbb{R}^2,$$

as well as $P = P_L \cap G_I \cap G_S$. We first show that the effective domain of $F \cap P$ locally coincides with the effective domain of $F \cap G$ even though P is potentially strictly smaller than G.

Claim 4. $F(x) \cap P \neq \emptyset$ for any $x \in \mathcal{D}$ in a neighborhood of x_0.

Fix $k \in K_\ell$ and a vector $z_0 \in F(x) \cap G$ such that L_k contains segments of length $\varepsilon > 0$ on both sides of $\pi_k(z_0)$. Because F is locally Lipschitz continuous, there exists a small neighborhood U_k of x_0 so that the Hausdorff distance of $F(x)$ and $F(x_0)$ is at most ε for any $x \in U_k$. Let N_k be an outward normal vector to G_k at L_k. Because G has a product structure, the vector N with $\pi_k(N) = N_k$ and $\pi_j(N) = 0$ for $j \neq k$ is the normal vector of a separating hyperplane H of $F(x_0)$ and G. In particular, $F(x)$ intersects G in a small neighborhood U_k' only if $F(x)$ also intersects H. Since $\pi_k(H \cap G) = L_k$, it follows that $F(x)$ intersects $L_k \times G_{-k} \subseteq P_k \times G_{-k}$ for any $x \in \mathcal{D} \cap U_k \cap U_k'$. Thus, the claim holds on the neighborhood $\bigcap_{k\in K_\ell} U_k \cap U_k'$ of x_0.

To show lower hemicontinuity, fix an open set V that intersects $F(x_0) \cap G = F(x_0) \cap P$. By definition, any $z_0 \in F(x_0) \cap P \cap V$ must satisfy $\pi_k(z_0) = z_*^k$ for $k \in K_s$. Moreover, since V is open, $F(x_0) \cap P \cap V$ must contain z_0 in the interior of G_I. Fix $z_0 \in \operatorname{int} G_I \cap V$.

Claim 5. For any $x \in \mathcal{D}$ and any $k \in K_s$, the projection $\pi_k(F(x) \cap P_L)$ is a rectangle.

Odd columns of A impose restrictions on the first dimension of each G_k, whereas even columns of A impose restrictions on the second dimension of each G_k. The condition on A imposes that these restrictions do not interact, and P_L is orthogonal to G_k.

Claim 5 implies that $\pi_k(F(x_n) \cap P)$ is a rectangle that moves tangentially around G_k as in Lemma F.5 with a singleton intersection $\pi_k(F(x_n) \cap P) \cap G_k = \{z_n^{(k)}\}$. It follows as in the proof of Lemma F.5 that $z_n^{(k)} \to \pi_k(z_0)$. Let now z_n denote the orthogonal projection of z_0 onto

$$Z_n := F(x_n) \cap P_L \cap \bigtimes_{k \in K_s} \{z_n^{(k)}\} \times \bigtimes_{k \notin K_s} \mathbb{R}^2.$$

We know that Z_n is non-empty because $F(x_n) \cap P \neq \emptyset$ for n sufficiently large by Claim 4 and each $z \in F(x_n) \cap P$ satisfies $\pi_k(z) = z_n^{(k)}$. By construction, $z_n \to z_0$ and, since $z_0 \in \operatorname{int} G_I \cap V$, we have $z_n \in F(x_n) \cap P \subseteq F(x_n) \cap G$ for large enough n. In particular, $F(x_n) \cap G \cap V \neq \emptyset$.

For an arbitrary sequence $(x_n)_{n \geq 1} \subseteq \mathcal{D}$ approaching x_0, we can write each $x_n = \eta_n y_n + (1 - \eta_n) x_0$ as a convex combination of $y_n \in \operatorname{ext} \mathcal{D}$ and x_0 since $\mathcal{D}$ is convex by Claim 1. If $(y_n)_{n \geq 1}$ converges to x_0, we construct $z_n \in F(y_n) \cap G$ that converges to z_0 as above. Then $z_n' := \eta_n z_n + (1 - \eta) z_0$ converges to z_0 as well and it must lie in $F(x_n) \cap G$ by Claim 1. If $(y_n)_{n \geq 1}$ does not converge to x_0, then η_n has to converge to 0. Thus, $z_n' := \eta_n z_n + (1 - \eta) z_0$ converges to z_0 for arbitrary $z_n \in F(y_n) \cap G$. □

F.1 Regularity of the state-transition optimality equation

In this appendix we prove Proposition 7.3 based on the preliminary results established in the previous subsection. It will be useful to introduce some additional notation.

- Let $\Psi^i_{y,\mathcal{A}^i_\diamond}(\alpha^{-i})$ and $\Psi^i_{y,\mathcal{A}_\diamond}$ denote the projections of $\Psi_{y,\mathcal{A}_\diamond}(\alpha)$ and $\Psi_{y,\mathcal{A}_\diamond}$ onto δ^i.
- Let $\Upsilon^i_{y,\mathcal{A}_\diamond}$ denote the set of all (α^{-i}, δ^i) that satisfy (10) for player i, all $a^i \in \mathcal{A}^i_\diamond$, and all $\tilde{a}^i \neq a^i$. Let $\Upsilon_{y,\mathcal{A}_\diamond} = \Upsilon^1_{y,\mathcal{A}_\diamond} \times \Upsilon^2_{y,\mathcal{A}_\diamond}$.

Moreover, when Assumption 2.(ii) is satisfied, let y_s denote the unique successor state of y so that $\delta\lambda(y, a) = \delta(y_s)\lambda_{y,y_s}(a)$. Label actions $\mathcal{A}^i(y) = \{0, 1\}$ so that $\lambda_{y,y_s}(1, a^{-i}) \geq \lambda_{y,y_s}(0, a^{-i})$ for any a^{-i} and each player i. We say λ is strictly monotone for player i if the inequality is strict. In that case, there is a unique $\delta^i_*(\alpha^{-i}, y_s)$ that incentivizes player i to mix against α^{-i}, i.e.,

$$g^i(y, 1, \alpha^{-i}) + \delta^i_*(\alpha^{-i}, y_s)\lambda_{y,y_s}(1, \alpha^{-i}) = g^i(y, 0, \alpha^{-i}) + \delta^i_*(\alpha^{-i}, y_s)\lambda_{y,y_s}(0, \alpha^{-i}). \qquad (31)$$

Lemma F.7. *Suppose that Assumption 2.(ii) holds with strictly monotone λ. Then:*

(i) $\delta^i_*(\alpha^{-i})$ *is either constant in* α^{-i} *or strictly monotone in* α^{-i} *with bounded derivative.*

(ii) $g^i(y, \alpha) + \delta^i_*(\alpha)\lambda(y, \alpha)$ *depends on* α *only through* α^{-i}.

(iii) $\Psi^i_{y,\{1\}}(\alpha^{-i}) = \{\delta^i \mid \delta^i(y_s) \geq \delta_*(\alpha^{-i}, y_s)\}$ *and* $\Psi^i_{y,\{0\}}(\alpha^{-i}) = \{\delta^i \mid \delta^i(y_s) \leq \delta_*(\alpha^{-i}, y_s)\}$.

Proof. Equation (31) implies that $\delta^i_*(\alpha^{-i})$ is a ratio of two affine functions, where the denominator is different from 0 for any α^{-i} by the monotonicity condition. It is easy to check that any such function satisfies statement (i). Player i is indifferent between both actions for $\delta^i_*(\alpha^{-i})$, hence any mixture α^i attains the same value $g^i(y,\alpha) + \delta^i_*(\alpha)\lambda(y,\alpha)$, showing statement (ii). Finally, since $\lambda_{y,y_s}(1,\alpha^{-i}) > \lambda_{y,y_s}(0,\alpha^{-i})$, it follows that player i is willing to play action 1 whenever $\delta^1 \geq \delta^1_*(\alpha^{-i})$ and action 0 whenever $\delta^1 \leq \delta^1_*(\alpha^{-i})$. □

Lemma F.8. *Fix y, $\mathcal{A}_\diamond \subseteq \mathcal{A}(y)$, and a family $\mathcal{W}$ of compact and convex payoff sets. Let $\Upsilon^i_{\mathcal{A}_\diamond}(w)$ denote the set of all (α,δ) with $\operatorname{supp}(\alpha) \subseteq \mathcal{A}_\diamond$ that satisfy (10) for player i and all $a^i \in \mathcal{A}^i_\diamond$ with $\mathcal{W}$-feasible δ. Suppose that either $\mathcal{A}_\diamond$ is a singleton or Assumption 2 is satisfied. Then:*

(i) $\Upsilon_{\mathcal{A}_\diamond}(w)$ is continuous on $\mathcal{K}_{y,\mathcal{A}_\diamond}(\mathcal{W})$ and locally Lipschitz continuous on its interior.

(ii) $\Upsilon^i_{\mathcal{A}_\diamond}(w)$ is continuous on its domain and locally Lipschitz continuous in its interior.

Proof. We prove both statements simultaneously because their proofs are virtually the same. It will be convenient to abbreviate by $\Delta_{\mathcal{A}_\diamond} := \Delta\mathcal{A}^1_\diamond \times \Delta\mathcal{A}^2_\diamond$ the set of mixed action profiles α with $\operatorname{supp}(\alpha) \subseteq \mathcal{A}_\diamond$. Let $\Lambda^i_y(a^i,\alpha^{-i})$ and $G^i_y(a^i,\alpha^{-i})$ denote the linear extensions of $\Lambda^i_y(a)$ and $G^i_y(a)$ in Section 7.1. Then $\Psi^i_{\mathcal{A}^i_\diamond}(\alpha^{-i})$ is the set of all δ^i that satisfy $G^i_y(a^i,\alpha^{-i}) + \delta^i\Lambda^i_y(a^i,\alpha^{-i}) \leq 0$ for all $a^i \in \mathcal{A}^i_\diamond$. To show local Lipschitz continuity, it will be convenient to reparametrize the space of incentives via $\tilde{\delta} = h(w) + r\delta$, where $h(w) \in \mathbb{R}^{2\times|\mathcal{Y}|}$ is the matrix containing w in each column. Let $h^i(w) = (w^i,\ldots,w^i)$ denote row i of $h(w)$. Denote by $\mathcal{Z}^i(w)$ the set of all $(\alpha,\tilde{\delta})$ that satisfy

$$rG^i_y(a^i,\alpha^{-i}) + \tilde{\delta}^i\Lambda^i_y(a^i,\alpha^{-i}) \leq h^i(w)\Lambda^i_y(a^i,\alpha^{-i}) \tag{32}$$

for each $a^i \in \mathcal{A}_\diamond$, and denote by $\mathcal{Z}(w)$ denote the set of all $(\alpha,\tilde{\delta})$ that satisfy (32) for each $a^i \in \mathcal{A}^i_\diamond$ and both players $i = 1,2$. Set $\tilde{\Upsilon}_{\mathcal{A}_\diamond}(w) := \mathcal{Z}(w) \cap \Delta_{\mathcal{A}_\diamond} \times \bigtimes_{y'} \mathcal{W}_{y'}$ and define $\tilde{\Upsilon}^i_{\mathcal{A}_\diamond}(w)$ analogously. Observe that $(\alpha,\delta) \in \Upsilon_{\mathcal{A}_\diamond}(w)$ if and only if $(\alpha, h(w) + r\delta) \in \tilde{\Upsilon}_{\mathcal{A}_\diamond}(w)$. In particular, the effective domains of $\tilde{\Upsilon}_{\mathcal{A}_\diamond}$ and $\Upsilon_{\mathcal{A}_\diamond}$ coincide and any continuity properties we establish for $\tilde{\Upsilon}_{\mathcal{A}_\diamond}$ also hold for $\Upsilon_{\mathcal{A}_\diamond}$.

Suppose first that either $\mathcal{A}_\diamond$ is a singleton or Assumption 2.(i) is satisfied. Then $\Lambda^i_y(a^i,\alpha^{-i})$ is independent of α^{-i}, hence $\mathcal{Z}^i(w)$ and $\mathcal{Z}(w)$ are polyhedral sets with affine right-hand side. Thus, Lemma F.4 shows that their intersection with the compact and convex set $\Delta_{\mathcal{A}_\diamond} \times \bigtimes_{y'} \mathcal{W}_{y'}$ is locally Lipschitz continuous on the interior of their domain. Let $F^i(w)$ and $F(w)$ denote the projections of $\mathcal{Z}^i(w)$ and $\mathcal{Z}(w)$, respectively, onto dimensions corresponding to δ. Since

$F^i(w)$ and $F(w)$ are polyhedra that satisfy the conditions of Lemma F.6, their intersection with $\bigtimes_{y'} \mathcal{W}_{y'}$ is continuous by Lemma F.6. Note that the $\tilde{\delta}$-sections $\mathcal{Z}(w \mid \tilde{\delta})$ of $\mathcal{Z}(w)$ are continuous in $\tilde{\delta}$ because they are polyhedra with a fixed orientation of hyperfaces. Therefore,

$$\tilde{\Upsilon}_{\mathcal{A}_\diamond}(w) = \left\{(\alpha, \tilde{\delta}) \;\middle|\; \tilde{\delta} \in F(w) \cap \bigtimes_{y'} \mathcal{W}_{y'}, \alpha \in \mathcal{Z}(w \mid \tilde{\delta})\right\}$$

is continuous on $\mathcal{K}_{y,\mathcal{A}_\diamond}(\mathcal{W})$. An analogous argument shows that $\tilde{\Upsilon}^i(w)$ is continuous.

Suppose next that Assumption 2.(ii) holds. We may assume without loss of generality that λ is strictly monotone as, otherwise, Assumption 2.(i) holds. It follows from Lemma F.7 that $\mathcal{Z}^i(w)$ is the set of all $(\alpha, \tilde{\delta})$ such that $\tilde{\delta}^i(y_s)$ is either equal to or bounded on one side by $w + r\delta_*^i(\alpha^{-i}, y_s)$. Because δ_*^i is monotone in α^{-i} by Lemma F.7, it follows that $F^i(w)$ is the set of all $\tilde{\delta}$ such that $\tilde{\delta}^i(y_s)$ is either equal to or bounded on one side by $w + r\delta_*^i(a^{-i}, y_s)$ for some pure action a^{-i}. The projection $F_{y_s}^i(w)$ of $F^i(w)$ onto $\delta(y_s)$ is thus a half-space with a locally Lipschitz continuous bound. Lemma F.5 shows that $F_{y_s}(w) \cap \mathcal{W}_{y_s}$ is locally Lipschitz continuous on the interior of the domain and continuous on the entire domain. Therefore, so is $F^i(w) \cap \bigtimes_{y'} \mathcal{W}_{y'} = F_{y_s}^i(w) \cap \mathcal{W}_{y_s} \bigtimes_{y' \neq y_s} \mathcal{W}_{y'}$. Because $\delta_*^i(\alpha^{-i})$ is monotone with bounded derivative by Lemma F.7, the $\tilde{\delta}$-sections $\mathcal{Z}^i(w \mid \tilde{\delta})$ of $\mathcal{Z}^i(w)$ are Lipschitz continuous. Therefore, $\tilde{\Upsilon}^i(w)$ is locally Lipschitz continuous on the interior of the domain and continuous on the entire domain. Note that $F(w) = F^1(w) \cap F^2(w)$, hence its projection $F_{y_s}(w)$ onto $\delta(y_s)$ are possibly unbounded rectangles such that all finite bounds are given by $w + r\delta_*^i(a^{-i}, y_s)$ for some pure action a^{-i}. The remainder of the argument is the same. □

Proof of Lemma 7.1. This is simply Statement (i) of Lemma F.8. □

Proof of Proposition 7.3. For any $w \in \bar{\mathcal{K}}_y(\mathcal{W}) \setminus \bar{\mathcal{S}}_y(\mathcal{W})$, the set $\mathcal{D}_y(w)$ is non-empty and bounded away from w since $\mathcal{D}_y(w)$ is closed. In particular, there are two tangents to $\mathcal{D}_y(w)$ through w and two solutions to (8). Outside of $\mathcal{K}_y(\mathcal{W})$ the map $\mathcal{D}_y$ is continuous, hence $\mathcal{D}_y(v)$ is bounded away from v close to w. Because $\mathcal{D}_y$ is uniformly bounded, the direction of the tangent T_w changes continuously in w, i.e., oriented solutions to (8) are continuously differentiable. Local Lipschitz continuity of (8) follows from Lemmas 7.1 and F.2. Uniqueness and continuity in initial conditions then follows from the Picard-Lindelöf theorem. □

F.2 Regularity of the optimality equation

In the optimality equation (9), the maximum is taken over all enforceable action profiles. However, none of our assumptions imply that all action profiles are enforceable. Let $\mathcal{V}_y^{\text{enf}}(\mathcal{W})$

denote the set of all payoff pairs that can be decomposed into (α, δ) for an enforceable action profile α and $\mathcal{W}$-feasible δ. Note that the requirement is not that δ from the decomposition enforces α, but that there exists some (β, δ') that enforce α. Define further

$$\underline{v}_y^{\text{enf},i}(\mathcal{W}) := \min_{\alpha,\beta,\delta} g^i(y,\alpha) + \delta^i \lambda(y,\alpha), \qquad \bar{v}_y^{\text{enf},i}(\mathcal{W}) := \max_{\alpha,\beta,\delta} g^i(y,\alpha) + \delta^i \lambda(y,\alpha), \tag{33}$$

where the minimum and maximum are taken over all enforceable α and all (β, δ) that enforce α with $\mathcal{W}$-feasible δ and $\beta^i = 0$. Assumption 1.(ii.c) guarantees that the minimization and maximization in (33) are taken over non-empty sets. If the $\mathcal{W}$-minmax profile against player i is enforceable, then $\underline{v}_y^{\text{enf},i}$ is player i's minmax payoff. Similarly, if the action profile that decomposes player i's highest $\mathcal{W}$-feasible payoff is enforceable, then $\bar{v}_y^{\text{enf}*,i}(\mathcal{W})$ coincides with that payoff. Let $\mathcal{V}_y^{\text{enf}*}(\mathcal{W})$ denote the set of all $w \in \mathcal{V}_y^{\text{enf}}(\mathcal{W})$ with $\underline{v}_y^{\text{enf},i}(\mathcal{W}) \leq w^i \leq \bar{v}_y^{\text{enf},i}(\mathcal{W})$. It follows exactly as in the proof of Lemma 5.2 that any enforceable strategy profile must attain values in $\mathcal{V}_y^{\text{enf}*}(\mathcal{W})$. The main result of this appendix is the following.

Proposition F.9. *Suppose that Assumption 1 holds and that players are either restricted to pure strategies or Assumption 2 holds. Then for any $w \in \mathcal{V}_y^{\text{enf}*}(\mathcal{W})$ and any $N \in S^1$, there exists an action profile α such that $\Xi_{y,\alpha}(w, N, \mathcal{W})$ is non-empty. Moreover,*

$$\kappa_y(w, N) := \max_{\alpha} \max_{(\beta,\delta)\in\Xi_{y,\alpha}(w,N,\mathcal{W})} \frac{2N^\top\big(g(y,\alpha) + \delta\lambda(y,\alpha) - w\big)}{r\|T^\top\beta\sigma(y)\|^2} \tag{34}$$

is locally Lipschitz continuous for (w, N) within $(\operatorname{int}\mathcal{V}_y^{\text{enf}*}(\mathcal{W}) \times S^1) \setminus \Gamma_y(\mathcal{W})$. In particular, solutions to (34) *are continuously differentiable and continuous in initial conditions.*

Remark F.1. The sufficient conditions in Lemma 4.5 imply that every action profile is enforceable, hence under these conditions $\mathcal{V}_y^{\text{enf}*}(\mathcal{W}) = \mathcal{V}_y^*(\mathcal{W})$.

We will break down the proof into several smaller results. Specifically, we will distinguish whether N is parallel to a coordinate axis, a so-called *coordinate direction*, or not. We do this for two reasons. First, no incentives can be provided to player i through tangential transfers when $N = \pm e_i$, hence local Lipschitz continuity requires a separate argument for coordinate directions. Second, and more importantly, the characterization of $\partial\mathcal{B}_y(\mathcal{W})$ relies on the perturbation argument described in Section 5.1. For the perturbation argument, we must allow continuation values after a transition to state y' to come from a set that is slightly larger than $\mathcal{W}_{y'}$. Since a perturbation argument for non-coordinate directions is sufficient, it is helpful to state regularity for coordinate and non-coordinate directions separately.

Specifically, for a family $\mathcal{W} = (\mathcal{W}_y)_{y\in\mathcal{Y}}$ of compact sets, call a family $\mathcal{L} = (\mathcal{L}_y)_{y\in\mathcal{Y}}$ of set-valued maps $\mathcal{L}_y : \mathbb{R}^2 \rightrightarrows \mathbb{R}^{2\times|\mathcal{Y}|}$ a *Lipschitz expansion* of $\mathcal{W}$ if

$$\mathcal{L}_y(w) = \left\{\delta \in \mathbb{R}^{2\times|\mathcal{Y}|} \,\middle|\, \min_{w_{y'}\in\mathcal{W}_{y'}} \left\|w + r\delta(y') - w_{y'}\right\| \leq h(w) \text{ for all } y' \in \mathcal{Y}\right\}$$

for a non-negative Lipschitz-continuous function $h : \mathbb{R}^2 \to \mathbb{R}$. A Lipschitz expansion is *concave* if h is concave. For $h \equiv 0$ this simply corresponds to $\mathcal{W}$-feasibility. Throughout the entire appendix, we keep the family $\mathcal{W}$, the Lipschitz expansion $\mathcal{L}$, and y fixed.

First, it will be helpful to reduce the number of variables in the maximization problem (9) by rewriting the optimal tangential transfers as a function of (α, δ, N). Let us again condition on the support $\mathcal{A}_\diamond$ of the players' mixture as follows. For any $\mathcal{A}_\diamond \subseteq \mathcal{A}(y)$, denote by

$$\Phi_{y,\mathcal{A}_\diamond}(\alpha, \delta, N) = \left\{\phi \in \mathbb{R}^{1\times d(y)} \,\middle|\, (T\phi, \delta) \text{ incentivizes each } a^i \in \mathcal{A}^i_\diamond \text{ against } \alpha^{-i} \text{ for } i = 1, 2\right\}$$

the set of all tangential transfers that make each player i indifferent among all actions $a^i \in \mathcal{A}^i_\diamond$ against α^{-i}, given δ, where T is the clockwise orthogonal direction to N. Denote by

$$\Upsilon_{y,\mathcal{A}_\diamond}(w, N, \mathcal{L}) := \left\{(\alpha, \delta) \in \Delta\mathcal{A}^1_\diamond \times \Delta\mathcal{A}^2_\diamond \times \mathcal{L}(w) \,\middle|\, \Phi_{y,\mathcal{A}_\diamond}(\alpha, \delta, N) \neq \emptyset\right\}$$

its effective domain. For any $(\alpha, \delta) \in \Upsilon_{y,\mathcal{A}_\diamond}(w, N, \mathcal{L})$, the optimal tangential transfers $\phi_{y,\mathcal{A}_\diamond}(\alpha, w, N)$ is the vector $\phi \in \Phi_{y,\mathcal{A}_\diamond}(\alpha, \delta, N)$ that minimizes $\|\phi\sigma(y)\|$.

Lemma F.10. *Suppose that Assumption 1.(ii) is satisfied. For any $\mathcal{A}_\diamond \subseteq \mathcal{A}(y)$:*

(i) $\phi_{y,\mathcal{A}_\diamond}$ *is locally Lipschitz continuous in* $(\alpha, \delta) \in \Upsilon_{y,\mathcal{A}_\diamond}(w, N, \mathcal{L})$ *and non-coordinate* N.

(ii) If δ^i *provides sufficient incentives for player* i *to play any* $a^i \in \mathcal{A}^i_\diamond$ *against* α^{-i}*, then* $\phi_y(\alpha, \delta, N)$ *is locally Lipschitz continuous in* N *at* $(\alpha, \delta, \pm e_i)$.

(iii) $\|\phi_{y,\mathrm{supp}(\alpha)}(\alpha, \delta, N)\| \leq \left\|\phi_{y,\mathcal{A}_\diamond}(\alpha, \delta, N)\right\|$ *for any* N *and any* $(\alpha, \delta) \in \Upsilon_{y,\mathcal{A}_\diamond}(w, N, \mathcal{L})$.

Proof. Note that $\Phi_{y,\mathcal{A}_\diamond}(\alpha, \delta, N)$ is the set of all ϕ that satisfy for $i = 1, 2$ and all $a^i \in \mathcal{A}^i_\diamond$,

$$\phi M^i_y(a^i, \alpha^{-i}) \leq -\frac{1}{T^i}\left(G^i_y(a^i, \alpha^{-i}) + \delta^i \Lambda^i_y(a^i, \alpha^{-i})\right) \tag{35}$$

As in Footnote 12, Assumption 1.(ii) implies that $M^i_y(a^i, \alpha^{-i})$ does not depend on α^{-i}. Therefore, $\Phi_{y,\mathcal{A}_\diamond}(\alpha, \delta, N)$ is a polyhedron with fixed orientation and "locally Lipschitz continuous right-hand side" for $N \neq \pm e_i$. Thus, statement (i) follows from Theorem 2.2 in Yen (1995).[16]

[16]In the notation of Yen's paper, Theorem 2.2 is applied to $c = 0$ and $D = \sigma(y)\sigma(y)^\top$, where D is positive

For the second statement, let $\Phi^i_{y,\mathcal{A}_\diamond}(\alpha,\delta)$ denote the set of all row vectors ϕ in $\mathbb{R}^{d(y)}$ that satisfy (35) for all $a^i \in \mathcal{A}_\diamond$ with $T^i = 1$. Let $\phi^i_{y,\mathcal{A}_\diamond}(\alpha,\delta)$ denote the vector $\phi^i \in \Phi^i_{y,\mathcal{A}_\diamond}(\alpha,\delta)$ that minimizes $\|\phi^i\sigma(y)\|$. If $(0,\delta^i)$ satisfies the enforceability constraint of player i, then the product structure implies that $\phi_{y,\mathcal{A}_\diamond}(\alpha,\delta,N) = \phi^{-i}_{y,\mathcal{A}_\diamond}(\alpha,\delta)/T^{-i}$. It follows again from Theorem 2.2 in Yen (1995) that $\phi^{-i}_{y,\mathcal{A}_\diamond}$ is Lipschitz continuous, hence so is $\phi_{y,\mathcal{A}_\diamond}(\alpha,\delta,N)$ for $N \neq \pm e_{-i}$.

The last statement follows because indifference among $\operatorname{supp}(\alpha^i)$ is potentially weaker than indifference among all $\mathcal{A}^i_\diamond$, hence $\phi_{y,\operatorname{supp}(\alpha)}$ is the minimum over a larger set than $\phi_{y,\mathcal{A}_\diamond}$ is. □

To substitute $\phi_{y,\mathcal{A}_\diamond}$ into the optimality equation, we must ensure that it is different from 0. Incentives from the public signal are not needed to enforce mixtures on $\mathcal{A}_\diamond$ on the set

$$\Gamma_{y,\mathcal{A}_\diamond}(\mathcal{L}) = \big\{(w,N) \,\big|\, \text{there exist } (\alpha,\delta) \in \Upsilon_{y,\mathcal{A}_\diamond}(w,N,\mathcal{L}) \text{ with } 0 \in \Phi_{y,\mathcal{A}_\diamond}(\alpha,\delta,N)\big\}.$$

Outside of $\Gamma_{y,\mathcal{A}_\diamond}(\mathcal{L})$, we can rewrite the optimality equation for support $\mathcal{A}_\diamond$ as

$$\kappa_{y,\mathcal{L},\mathcal{A}_\diamond}(w,N) := \max_{(\alpha,\delta)\in\Upsilon_{y,\mathcal{A}_\diamond}(w,N,\mathcal{L})} \frac{2N^\top\big(g(y,\alpha)+\delta\lambda(y,\alpha)-w\big)}{r\big\|\phi_{y,\mathcal{A}_\diamond}(\alpha,\delta,N)\sigma(y)^2\big\|} \vee 0. \tag{36}$$

On the set $E_{y,\mathcal{A}_\diamond}(\mathcal{L}) = \big\{(w,N) \,\big|\, \Upsilon_{y,\mathcal{A}_\diamond}(w,N,\mathcal{L}) \neq \emptyset\big\}$ that maximization in (36) is taken over a non-empty set. We are now ready to establish local Lipschitz continuity of $\kappa_{y,\mathcal{A}_\diamond}$.

Lemma F.11. *Suppose Assumption 1.(ii) holds. For any Lipschitz expansion $\mathcal{L}$ of $\mathcal{W}$ and any $\mathcal{A}_\diamond \subseteq \mathcal{A}(y)$ such that there exists at least one enforceable action profile α with* $\operatorname{supp}(\alpha) = \mathcal{A}_\diamond$*:*

(i) $E_{y,\mathcal{A}_\diamond}(\mathcal{L})$ contains any pair (w,N) for non-coordinate direction N.

(ii) $\kappa_{y,\mathcal{L},\mathcal{A}_\diamond}(w,N)$ is locally Lipschitz continuous at $(w,N) \notin \Gamma_{y,\mathcal{A}_\diamond}(\mathcal{L}) \cup (\mathbb{R}^2 \times \{\pm e_i\})$.

Proof. Let $\Delta^{\text{enf}}_{\mathcal{A}_\diamond}$ be the closure of the set of all enforceable action profiles α with $\operatorname{supp}(\alpha) = \mathcal{A}_\diamond$. Fix any payoff pair w, any non-coordinate direction N, any $\alpha \in \Delta^{\text{enf}}_{\mathcal{A}_\diamond}$, and any δ in $\mathcal{L}_y(w)$. Let $(\alpha_n)_{n\geq 0} \subseteq \Delta^{\text{enf}}_{\mathcal{A}_\diamond}$ be a sequence with support $\mathcal{A}_\diamond$ that approximates α. By Assumption 1.(i), there exists β_n such that (β_n,δ) enforces α_n for each n. Lemma 4 in Bernard and Frei (2016) applied to payoff function $\tilde{g}(y,\alpha_n) = g(y,\alpha_n) + \delta\lambda(y,\alpha_n)$ shows that there exists ϕ_n such that $(T\phi_n,\delta)$ enforces α_n. In particular, $\Phi_{y,\mathcal{A}_\diamond}(\alpha_n,\delta,N) \neq \emptyset$. Because the domain of $\Phi_{y,\mathcal{A}_\diamond}$ is closed, it follows that $\Phi_{y,\mathcal{A}_\diamond}(\alpha,\delta,N) \neq \emptyset$. In particular, $(\alpha,\delta) \in \Upsilon_{y,\mathcal{A}_\diamond}(w,N,\mathcal{L})$, hence $(w,N) \in E_{y,\mathcal{A}_\diamond}(\mathcal{L})$. Moreover, because α and δ were arbitrary, this shows that $\Upsilon_{y,\mathcal{A}_\diamond}(\alpha,w,\mathcal{L}) = \Delta^{\text{enf}}_{\mathcal{A}_\diamond} \times \mathcal{L}_y(w)$, which is locally Lipschitz continuous in (α,w,N). Together

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definite as required because $\sigma(y)$ has full rank.

with Lemmas F.2 and F.10, this shows that $\kappa_{y,\mathcal{A}_\diamond}(w, N, \mathcal{L})$ is locally Lipschitz continuous in w and non-coordinate N outside of $\Gamma_{y,\mathcal{A}_\diamond}(\mathcal{L})$. □

Corollary F.12. *For any concave Lipschitz expansion $\mathcal{L}$ of $\mathcal{W}$, the function*

$$\kappa_{y,\mathcal{L}}(w, N) = \max_{\mathcal{A}_\diamond \subseteq \mathcal{A}(y)} \kappa_{y,\mathcal{L},\mathcal{A}_\diamond}(w, N)$$

is locally Lipschitz continuous at (w, N) outside of $\Gamma_y(\mathcal{L}) \cup (\mathbb{R}^2 \times \{\pm e_i\})$. In particular, solutions to (34) *are continuously differentiable and continuous in initial conditions.*

Proof. For any $\mathcal{A}_\diamond \subseteq \mathcal{A}(y)$ that does not support an enforceable action profile $\kappa_{y,\mathcal{L},\mathcal{A}_\diamond} \equiv -\infty$, hence it cannot contribute to the maximum. For any other $\mathcal{A}_\diamond \subseteq \mathcal{A}(y)$, the effective domain is $\mathbb{R}^2 \times (S^1 \setminus \{\pm e_i\})$ by Lemma F.11, hence $\kappa_{y,\mathcal{L}}$ maximizes over the sames sets $\mathcal{A}_\diamond$ for any (w, N). Because each $\kappa_{y,\mathcal{L},\mathcal{A}_\diamond}$ is locally Lipschitz continuous, the maximum over finitely many is locally Lipschitz continuous as well. The last statement is the Picard-Lindelöf theorem. □

Corollary F.12 provides sufficient regularity for the perturbation argument discussed in Section 5.1. For the remainder of this appendix, we establish local Lipschitz continuity of the optimality equation for the trivial Lipschitz expansion $\mathcal{L}_0$ of $\mathcal{W}$ with $h \equiv 0$.

Lemma F.13. *Suppose Assumption 1 holds and that either players are restricted to pure strategies or Assumption 2 also holds. For any coordinate direction N_0, $\kappa_{y,\mathcal{L}_0}(w, N)$ is locally Lipschitz continuous at any $(w, N_0) \in \bigcup_{\mathcal{A}_\diamond} \operatorname{int} E_{y,\mathcal{A}_\diamond}(\mathcal{L}_0) \setminus \Gamma_{y,\mathcal{A}_\diamond}(\mathcal{L}_0)$ with $\kappa_{y,\mathcal{L}_0}(w, N_0) > 0$.*

Proof. Fix some support $\mathcal{A}_\diamond \subseteq \mathcal{A}(y)$ and a coordinate direction $N_0 = \pm e_i$. Since ${N_0}^\top \beta = 0$ requires $\beta^i = 0$, incentives for player i must be be provided through state transitions. By Assumption 1.(ii), player $-i$'s incentives can be provided entirely through the public signal for any δ^{-i}, which implies $\Upsilon_{y,\mathcal{A}_\diamond}(w, N_0, \mathcal{L}_0) = \Upsilon^i_{y,\mathcal{A}_\diamond}(w, \mathcal{W})$. Therefore, local Lipschitz continuity of $\Upsilon_{y,\mathcal{A}_\diamond}(w, N_0, \mathcal{L}_0)$ follows from Lemma F.8. Lemmas F.2 and F.10 now imply that

$$\kappa^i_{\mathcal{A}_\diamond}(w, N) := \max_{(\alpha,\delta) \in \Upsilon^i_{y,\mathcal{A}_\diamond}(w,\mathcal{W})} \frac{2N^\top(g(y,\alpha) + \delta\lambda(y,\alpha) - w)}{r\left\|\phi_{y,\mathcal{A}_\diamond}(\alpha, \delta, N)\sigma(y)^2\right\|} \vee 0$$

is locally Lipschitz continuous at $(w, N_0) \in \operatorname{int} E_{y,\mathcal{A}_\diamond}(\mathcal{L}_0) \setminus \Gamma_{y,\mathcal{A}_\diamond}(\mathcal{L}_0)$ with coordinate N_0. Because $\kappa^i_{\mathcal{A}_\diamond}$ coincides with $\kappa_{y,\mathcal{L}_0,\mathcal{A}_\diamond}$ for coordinate directions, this shows local Lipschitz continuity of $\kappa_{y,\mathcal{L}_0,\mathcal{A}_\diamond}$ in w. Local Lipschitz continuity of $\kappa_{y,\mathcal{L}^0}$ in w follows as the finite maximum over such functions.

Local Lipschitz continuity in N will follow once we establish that $(\alpha, \delta) \notin \Upsilon^i_{y,\mathrm{supp}(\alpha)}(w, \mathcal{W})$ cannot attain the maximum in (36) in a neighborhood of (w, N_0) with $\kappa_{y,\mathcal{L}_0}(w, N_0) > 0$. Then, $\kappa_{y,\mathcal{L}^0}$ must locally coincide with $\max_{\mathcal{A}_\diamond} \kappa^i_{\mathcal{A}_\diamond}$. To that end, let $\phi^i_y(\alpha, \delta)$ denote the vector ϕ^i that minimizes $\|\phi^i \sigma(y)\|$ among all vectors ϕ^i such that incentives $(T\phi^i, \delta^i)$ player i to play α^i against α^{-i}. The product structure of the public signal implies that

$$\|\phi_y(\alpha, \delta, N)\sigma(y)\| = \|\phi^1_y(\alpha, \delta)\sigma(y)\|/|N^2| + \|\phi^2_y(\alpha, \delta)\sigma(y)\|/|N^1|.$$

It follows that $\|\phi_y(\alpha, \delta, N)\sigma(y)\| \to \infty$ as $N \to N_0$ for any $(\alpha, \delta) \notin \Upsilon^i_{y,\mathrm{supp}(\alpha)}(w, \mathcal{W})$. Let $f(\alpha, \delta, N)$ denote the function maximized in the optimality equation (34). Since the numerator of f is uniformly bounded, it follows that $f(\alpha, \delta, N) \to 0$ as $N \to N_0$. Conversely, local Lipschitz continuity of each $\kappa^i_{\mathcal{A}_\diamond}$ implies that there exist $(\alpha, \delta) \in \Upsilon^i_{y,\mathrm{supp}(\alpha)}(w, \mathcal{W})$ that attain values arbitrarily close to $\kappa_{y,\mathcal{L}_0}(w, N_0) > 0$ in a small neighborhood of (w, N_0), hence $\kappa_{y,\mathcal{L}}$ locally coincides with $\max_{\mathcal{A}_\diamond} \kappa^i_{\mathcal{A}_\diamond}$. In particular, $\kappa_{y,\mathcal{L}^0}$ is locally Lipschitz continuous. □

Lemma F.14. *Suppose Assumption 1 holds. For any Lipschitz expansion $\mathcal{L}$ and any (w, N) in* $\mathrm{int}\, \mathcal{V}^{\mathrm{enf}*}_y(\mathcal{W}) \times S^1$, *there exist $(\alpha, \delta) \in \Upsilon_{y,\mathrm{supp}(\alpha)}(w, N, \mathcal{L})$ with $N^\top(g(y, \alpha) + \delta\lambda(y, \alpha)) > 0$.*

Proof. Fix a payoff pair $w_0 \in \mathrm{int}\, \mathcal{V}^{\mathrm{enf}*}_y(\mathcal{W})$. For any non-coordinate N, let w_N denote some $v \in \mathcal{V}^{\mathrm{enf}}_y(\mathcal{W})$ that maximizes $N^\top v$. By definition, w_N is decomposed by an enforceable action profile α_N and incentives δ_N. Let δ_0 denote the $\mathcal{W}$-feasible incentives at w_0 that maximize $N^\top\delta(y')$ for each y'. This implies $N^\top(w_0 + r\delta(y')) \geq N^\top(w_N + r\delta_N(y'))$ for each y' and, hence

$$N^\top(\delta_0(y') - \delta_N(y')) \geq \frac{1}{r} N^\top(w_N - w_0) > 0. \tag{37}$$

Because $\lambda_{y,y'}(\alpha_N) \geq 0$, it follows that

$$N^\top(g(y, \alpha_N) + \delta_0\lambda(y, \alpha_N) - w_0) > N^\top(g(y, \alpha_N) + \delta_N\lambda(y, \alpha_N) - w_N) = 0. \tag{38}$$

Since δ_0 is $\mathcal{W}$-feasible, it lies in $\mathcal{L}^0_y(w_0) \subseteq \mathcal{L}_y(w_0)$. By Assumption 1.(ii.b), there exists β such that (δ_0, β) restricted enforces α_N, which shows that $\kappa_{y,\mathcal{L}}(w, N) > 0$.

Suppose next that N is coordinate, and suppose $N = e_i$ for the sake of specificity. Let (α_N, δ_N) attain $\bar{v}^{\mathrm{enf},i}_y(\mathcal{W})$, denote $w_N = g(y, \alpha_N) + \delta_N\lambda(y, \alpha_N)$, and set $w_t := tw_N + (1-t)w_0$. For any $t \in [0, 1]$, let δ_t be defined by $w_t + r\delta_t(y') = w_N + r\delta_N(y')$ for all states y', and let a^i_t denote player i's action that maximizes $\ell(t, a^i) := g^i(y, a^i, \alpha^{-i}_N) + \delta^i_t\lambda(y, a^i, \alpha^{-i}_N)$. Because δ_t is affine in t, so is $\ell(t, a^i)$ for each a^i. In particular, $\ell(t) := \max_{a^i} \ell(t, a^i)$ is continuous and the maximizer a_t^i changes only finitely many times. On each segment where a_t^i is constant, it follows as in (37) and (38) that $\ell(t)$ is decreasing in t. Since $\delta_1 = \delta_N$, it follows that $\ell(1) = 0$, hence $\ell(0) > 0$. By Assumption 1.(ii.c), there exist α^{-i} and β with $\beta^i = 0$ such that (δ_w, β) enforces $(\alpha^i, \alpha_N^{-i})$. Because α^i is incentivized by δ_w^i alone, any pure action in its support must maximize $\ell(0, a^i)$. Thus, α^i must satisfy $g^i(y, \alpha^i, \alpha_N^{-i}) + \delta_t^i \lambda(y, \alpha^i, \alpha_N^{-i}) = \ell(0) > 0$. □

Proof of Proposition F.9. Observe that $\kappa_y(w, N)$ defined in (34) coincides with $\kappa_{y,\mathcal{L}^0}(w, N)$ defined in Corollary F.12. While $\kappa_{y,\mathcal{L}^0}(w, N)$ may maximize over the same action profile multiple times, once for each $\mathcal{A}_\diamond$ that contains $\operatorname{supp}(\alpha)$, it follows from Lemma F.10.(iii) that largest value is attained for $\mathcal{A}_\diamond = \operatorname{supp}(\alpha)$. Local Lipschitz continuity for non-coordinate directions is established in Corollary F.12. Local Lipschitz continuity at coordinate directions follows from Lemmas F.13 and F.14. Finally, fix any $w_* \in \mathcal{V}_y^{\mathrm{enf}*}(\mathcal{W})$ and N. Let $(w_n)_{n\geq 0} \subseteq \operatorname{int} \mathcal{V}_y^{\mathrm{enf}*}(\mathcal{W})$ approximate w. Lemma F.14 implies that there exist $(\alpha_n, \delta_n) \in \Upsilon_{y,\operatorname{supp}(\alpha_n)}(w_n, N, \mathcal{L}^0)$ with $N^\top(g(y, \alpha_n) + \delta_n \lambda(y, \alpha_n) - w_n) > 0$. By passing to a subsequence, we can assume that the support $\mathcal{A}_\diamond$ along the entire sequence is constant. By passing to a further subsequence, we may assume that α_n and δ_n converge to limits α and δ, respectively. Since we have established in the proofs of Lemmas F.11 and F.13 that $\Upsilon_{y,\mathcal{A}_\diamond}(w, N, \mathcal{L}^0)$ is continuous in w, it follows that $(\alpha, \delta) \in \Upsilon_{y,\mathcal{A}_\diamond}(w_*, N, \mathcal{L}^0)$, showing that $\Xi_{y,\alpha}(w, N, \mathcal{W}) \neq \emptyset$. □

G Characterizations of $\mathcal{S}_y(\mathcal{W})$ and $\mathcal{K}_y(\mathcal{W})$

G.1 Local characterization of stationary payoffs

Proof of Lemma 6.2. For the first statement, note that $\mathcal{D}_{\mathcal{A}_\diamond}(w)$ is compact as the continuous image of the compact set $\Upsilon_{\mathcal{A}_\diamond}(w)$. If $\mathcal{A}_\diamond = \{a\}$ is a singleton, then $\mathcal{D}_{\mathcal{A}_\diamond}(w)$ is convex as the image of the convex set $\Psi_{\mathcal{A}_\diamond}(w)$ under the affine map $g(y, a) + \delta\lambda(y, a)$. The second statement follows from Lemma F.8 since $\mathcal{D}_{\mathcal{A}_\diamond}(w)$ is the continuous image of the continuous set $\Upsilon_{\mathcal{A}_\diamond}(w)$.

For the third statement, fix an arbitrary payoff pair $w_0 \in \mathcal{K}_{y,\mathcal{A}_\diamond}(\mathcal{W})$ and two directions N and N' with $N^\top N'$. Abbreviate $w_\varepsilon := w_0 + \varepsilon N'$, and let $F(w_\varepsilon)$ denote the set of all $\mathcal{W}$-feasible δ at w_ε. Fix an arbitrary action profile α supported on $\mathcal{A}_\diamond$ and set

$$f_\alpha(\varepsilon) := \max_{\delta \in \Psi_{\mathcal{A}_\diamond}(\alpha) \cap F(w_\varepsilon)} N^\top \delta\lambda(y, \alpha).$$

We will show that there exists $\varepsilon_0 > 0$ so that $f_\alpha(\varepsilon)$ is non-increasing for all $\varepsilon > \varepsilon_0$ and any α. In particular, then so is $\max_\alpha f_\alpha(\varepsilon)$. Let us view incentives δ as vectors $(\delta^1, \delta^2) \in$

$\mathbb{R}^{2|\mathcal{Y}|}$. In this notation, $f_\alpha(\varepsilon)$ maximizes the convex set $\Psi_{\mathcal{A}_\diamond}(\alpha) \cap F(w_\varepsilon)$ in direction $N_\lambda := (N^1\lambda(y,\alpha), N^2\lambda(y,\alpha))$. It will be convenient to denote by N_w the vector that contains $N'^{,1}$ in the first $|\mathcal{Y}|$ dimensions and $N'^{,2}$ in the latter dimensions so that a shift of w_0 by $\varepsilon N'$ corresponds to a shift of the feasible set by $-\varepsilon N_w$.

Suppose first that only feasibility constraints bind at w_0, hence also at w_ε for ε sufficiently small by continuity. Then any maximizing incentives δ_ε at w_ε must be maximally feasible, i.e., $N^\top(w_\varepsilon + r\delta_\varepsilon(y')) = \max_{v\in\mathcal{W}_{y'}} N^\top v$ for each y'. It follows that $\delta_\varepsilon = \delta_0 - \varepsilon N_w/r$, hence

$$N_\lambda{}^\top\delta_\varepsilon = N_\lambda{}^\top\delta_0 - \frac{\varepsilon}{r}N_w{}^\top N_\lambda \le N_\lambda{}^\top\delta_0$$

shows that f is locally non-increasing. Note that the inequality is strict unless $\lambda(y,\alpha) = 0$. Suppose next that no feasibility constraint binds in direction N', that is, there exists sufficiently small $\varepsilon > 0$ such that $w + \varepsilon N' + r\delta_w(y') \in \mathcal{W}_{y'}$ for each y'. Then δ_0 must maximize $N_\lambda{}^\top\delta$ at w_ε for $\varepsilon > 0$ sufficiently small, hence f is locally constant. Finally, suppose that both feasibility and enforceability constraints bind. Since both $F(w_0)$ and $\Psi_{\mathcal{A}_\diamond}(\alpha)$ are convex, there must exist outward normal vectors N_F and N_E to $F(w_0)$ and $\Psi_{\mathcal{A}_\diamond}(\alpha)$, respectively, and constants $\gamma_F, \gamma_E > 0$ such that $\gamma_F N_F + \gamma_E N_E = N_\lambda$; see Figure 17. Let H denote the supporting hyperplane to $F(w_0)$ at δ_0. Since the entire feasible set shifts by $-\varepsilon N_w/r$, so does the supporting hyperplane. It follows that any feasible $\delta \in F(w_\varepsilon)$ satisfies $N_F{}^\top\delta \le N_F{}^\top\delta_0 - \frac{\varepsilon}{r}N_w{}^\top N_F$. Any $\delta \in \Psi_{\mathcal{A}_\diamond}(\alpha)$ satisfies $N_E{}^\top\delta \le N_E{}^\top\delta_0$. In particular,

$$N_\lambda{}^\top\delta = \gamma_E N_E{}^\top\delta + \gamma_F N_F{}^\top\delta \le \gamma_E N_E{}^\top\delta_0 + \gamma_F N_F{}^\top\delta_0 - \frac{\varepsilon}{r}N_w{}^\top N_F.$$

Let $N_F(\varepsilon)$ denote a normal vector to $F(w_\varepsilon)$ at the intersection with $\Psi_{\mathcal{A}_\diamond}(\alpha)$. Note that $N_w{}^\top N_F(\varepsilon)$ is concave because $F(w_\varepsilon)$ is convex: the more $F(w_\varepsilon)$ is pulled inside of $\Psi_{\mathcal{A}_\diamond}(\alpha)$, the less aligned $N_F(\varepsilon)$ and N_w become; see Figure 17. To see that f is eventually non-decreasing, note that $F(w_\varepsilon)$ is a compact set that travels through the convex set $\Psi_{\mathcal{A}_\diamond}(\alpha)$. Thus either $F(w_\varepsilon)$ leaves $\Psi_{\mathcal{A}_\diamond}(\alpha)$ for some ε_0, at which point $N_w{}^\top N_F(\varepsilon_0) < 0$ has to hold, or $F(w_\varepsilon)$ intersects $\Psi_{\mathcal{A}_\diamond}(\alpha)$ forever. Because $\Psi_{\mathcal{A}_\diamond}(\alpha)$ consists of finitely many hyperfaces, if the sets intersect forever, eventually $\Psi_{\mathcal{A}_\diamond}(\alpha) \cap F(w_\varepsilon)$ is a translation in the direction N_w and f is decreasing. Assumption 2 guarantees that the orientation of the hyperfaces are constant, hence α impacts only the location of the hyperfaces. However, those changes are uniformly bounded, hence ε_0 can be chosen uniformly across α. □

Proof of Lemma 6.3. Since $\mathcal{D}_{\mathcal{A}_\diamond}(w)$ is closed-valued, any curve $\mathcal{C}$ that satisfies $w \in \partial\mathcal{D}_{\mathcal{A}_\diamond}(w)$

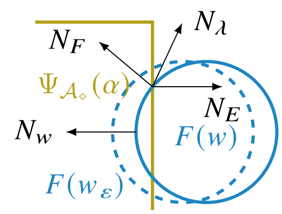

Figure 17: Local change in $\max_\delta N_\lambda^\top \delta$ when feasibility and enforceability constraints bind.

for all $w \in \mathcal{C}$ is contained in $\mathcal{S}_{y,\mathcal{A}_\diamond}(\mathcal{W})$. The exterior boundary of any connected component of $\mathcal{S}_{y,\mathcal{A}_\diamond}(\mathcal{W})$ is a closed curve. Suppose there exists $w \in \partial\mathcal{S}_{y,\mathcal{A}_\diamond}(\mathcal{W})$ in the interior of $\mathcal{D}_{\mathcal{A}_\diamond}(w)$. Then continuity of $\mathcal{D}_{\mathcal{A}_\diamond}$ implies that for any neighborhood V of w, there exists a neighborhood U of w such that $V \subseteq \mathcal{D}_{\mathcal{A}_\diamond}(v)$ for any $v \in U$. In particular, $v \in \mathcal{D}_{\mathcal{A}_\diamond}(v)$ for any $v \in U \cap V$, showing that $w \in \operatorname{int} \mathcal{S}_{y,\mathcal{A}_\diamond}(\mathcal{W})$. □

Proof of Lemma 6.4. For any $(\alpha, \delta) \in \Upsilon_{\mathcal{A}_\diamond}$, player i is indifferent between any actions in $\mathcal{A}^i_\diamond$. Thus, we can fix arbitrary $a^i_0 \in \mathcal{A}^i_\diamond$ and express the decomposition of player i's payoff as

$$v_i(\alpha^{-i}, \delta^i) := g^i(y, a^i_0, \alpha^{-i}) + \delta^i \lambda(y, a^i_0, \alpha^{-i}).$$

We first show monotonicity of the pre-image under the decomposition.

Claim 6. For any α^{-i} with $\lambda(y, a^i_0, \alpha^{-i}) \neq 0$, the set $H_i(v^i_0, \alpha^{-i}) := \left\{\delta^i \,\middle|\, v_i(\alpha^{-i}, \delta^i) = v^i_0\right\}$ is a hyperplane with constant orientation that moves monotonically in α^{-i}.

Suppose first that Assumption 2.(ii) is satisfied. Then $H_i(v_0, \alpha^{-i})$ consists of only a single point $\delta^i = (v^i_0 - g^i(y, a^i_0, \alpha^{-i}))/\lambda^i(y, a^i_0, \alpha^{-i})$, which is monotone in α^{-i} as a fractional linear function as in the proof of Lemma F.7. Suppose next that Assumption 2.(i) and Condition (ii) of Lemma 4.5 are satisfied with linear λ. Linearity of λ imposes that there exists a vector λ_1 such that $\lambda(y, a^i_0, \alpha^{-i}) = a^i_0 \bar{\alpha}^{-i} \lambda_1$, where $\bar{\alpha}^{-i}$ is the expected action of α^{-i}. Thus, $H_i(v_0, \alpha^{-i})$ is the set of all δ^i such that $\delta^i \lambda_1 = (v^i_0 - g^i(y, a^i_0, \alpha^{-i}))/(a^i_0 \bar{\alpha}^{-i})$. It follows as in the proof of Lemma 4.5 that in any enforceable mixed action profile, each player i must place weight only on two adjacent pure actions. Thus, the ratio is again monotone as a fractional linear function.

Claim 7. Let $\Upsilon^i_{\mathcal{A}_\diamond}(w)$ be the projection of $\Upsilon_{\mathcal{A}_\diamond}(w)$ onto coordinates (α^{-i}, δ^i). The incentive-feasible pre-image $v_i^{-1}(v^i_0) = \left\{(\alpha^{-i}, \delta^i) \in \Upsilon^i_{\mathcal{A}_\diamond}(w) \,\middle|\, \delta^i \in H_i(v^i_0, \alpha^{-i})\right\}$ is path-connected.

Fix any $(\alpha^{-i}_0, \delta^i_0), (\alpha^{-i}_1, \delta^i_1) \in v_i^{-1}(v^i_0)$ and set $\alpha^{-i}_t := \max\{2t, 1\}\alpha^{-i}_1 + \min\{1 - 2t, 0\}\alpha^{-i}_0$. By Claim 6, the set $H(t) := \{\alpha^{-i}_t\} \times H_i(v^i_0, \alpha^{-i}_t)$ is a hyperplane with fixed orientation that moves monotonically in t. Since $H(0)$ and $H(1/2) = H(1)$ intersect the convex set $\Upsilon^i_{\mathcal{A}_\diamond}(w)$, monotonicity implies that $H(t)$ intersects $\Upsilon_{\mathcal{A}_\diamond}(w)$ for any $t \in (0, \frac{1}{2})$. Next, observe that $\Upsilon^i_{\mathcal{A}_\diamond}(w)$ is a polyhedron since the projection of $\mathcal{W}$-feasibility yields the polyhedral constraints $\underline{w}^i_{y'} \leq w^i + r\delta^i(y') \leq \bar{w}^i_{y'}$, where $\underline{w}^i_{y'}$ and $\bar{w}^i_{y'}$ are player i's lowest and highest payoff in $\mathcal{W}_{y'}$, respectively. It follows that $H(t) \cap \Upsilon^i_{\mathcal{A}_\diamond}(w)$ is a polyhedron with constant orientation of hyperfaces and "continuous right-hand side," hence it is a continuous set-valued map. For any $t \in (0, \frac{1}{2}]$, let δ^i_t denote the projection of δ^i_0 onto $H(t) \cap \Upsilon^i_{\mathcal{A}_\diamond}(w)$, which is continuous. For $t \in (\frac{1}{2}, 1)$, set $\delta^i_t = (2t-1)\delta^i_1 + (2-2t)\delta^i_{1/2}$, which is continuous and lies in $H(1) \cap \Upsilon^i_{\mathcal{A}_\diamond}(w)$ by convexity. Thus, $(\alpha^{-i}_t, \delta^i_t)$ is a continuous path from $(\alpha^{-i}_0, \delta^i_0)$ to $(\alpha^{-i}_1, \delta^i_1)$.

Claim 8. Define the coordinate sections of $\mathcal{D}_{\mathcal{A}_\diamond}$, given v^{-i}_0 and conditional on δ^{-i}_0, as

$$\mathcal{D}_{\mathcal{A}_\diamond}(w \mid v^{-i}_0, \delta^{-i}_0) = \left\{ v \;\middle|\; \exists (\alpha, \delta) \in \Upsilon_{\mathcal{A}_\diamond}(w) \text{ with } v = v(\alpha, \delta), v^{-i} = v^{-i}_0, \text{ and } \delta^{-i} = \delta^{-i}_0 \right\}.$$

Under above conditions, $\mathcal{D}_{\mathcal{A}_\diamond}(w \mid v^{-i}_0, \delta^{-i}_0)$ is a continuous compact- and convex-valued map.

As in the proof of Lemma 5.5, $\Upsilon^i_{\mathcal{A}_\diamond}$ is a polyhedron. For any fixed δ^{-i}, let $F_i(w \mid \delta^{-i})$ denote the set of all δ^i such that (δ^i, δ^{-i}) is $\mathcal{W}$-feasible at w. It is the polyhedron defined by

$$\ell^i_{y'}(w^{-i} + r\delta^{-i}(y')) \leq w^i + r\delta^i(y') \leq u^i_{y'}(w^{-i} + r\delta^{-i}(y'))$$

for parametrizations $\ell^i_{y'}$ and $u^i_{y'}$ of the lower and upper frontier of $\mathcal{W}_{y'}$, respectively. Since $\ell^i_{y'}$ and $u^i_{y'}$ are continuous, $\Upsilon^i_{\mathcal{A}_\diamond}(w, \delta^{-i}) := \Upsilon^i_{\mathcal{A}_\diamond} \cap \Delta(\mathcal{A}^{-i}_\diamond) \times F_i(w \mid \delta^{-i})$ is a polyhedron with constant orientation of hyperfaces and continuous right-hand side. In particular, it is a continuous compact- and convex-valued map. Therefore, so is $\mathcal{D}_{\mathcal{A}_\diamond}(w \mid v^{-i}_0, \delta^{-i}_0)$ as the image of $\Upsilon^i_{\mathcal{A}_\diamond}(w, \delta^{-i}_0)$ under the continuous map v_i. Finally, the coordinate section

$$\mathcal{D}_{\mathcal{A}_\diamond}(w \mid v^{-i}_0) = \bigcup_{(\alpha^i_0, \delta^{-i}_0) \in v^{-1}_{-i}(v^{-i}_0)} \mathcal{D}_{\mathcal{A}_\diamond}(w \mid v^{-i}_0, \delta^{-i}_0)$$

is path-connected as the continuous image of a path-connected set by Claims 7 and 8. Since the coordinate section is one-dimensional, path-connectedness coincides with convexity.

To show that $\mathcal{S}_{y,\mathcal{A}_\diamond}(\mathcal{W})$ has convex coordinate sections, we extend the above argument to include the decomposed payoff pair. Specifically, reparametrize $\tilde{\delta} = h(w) + r\delta$ as in the proof of Lemma F.8, where $h(w)$ is the matrix that contains w in each column. The set $\mathcal{Z}_{\mathcal{A}_\diamond}$ of all $(\alpha, \tilde{\delta}, w)$ that satisfy the transformed enforceability constraint (32) is a polyhedron with constant orientation of hyperfaces, and the remainder of the argument is analogous. □

G.2 Simplifications in specific settings

Proof of Lemma 6.1. Let $w_*(\alpha) := g(y,\alpha) + \delta_*(\alpha)\lambda(y,\alpha)$ denote the unique payoff decomposed by α and δ_*, and by $v_*(\alpha) := w_*(\alpha) + r\delta_*(\alpha, y_s)$ the continuation payoff reached from $w_*(\alpha)$ after a state transition. It follows from (31) that $w_*^i(\alpha)$ and $v_*^i(\alpha)$ do not depend on α^i.

Claim 9. The function $v_*^i(\alpha)$ is either concave or convex in α^{-i} for $i = 1, 2$.

Equation (31) implies that $\delta_*^i(\alpha^{-i})$ is a ratio of two affine functions, where the denominator is different from 0 for any α^{-i} by the monotonicity condition. Thus, there exist constants $c_1, \dots, c_8$ with $c_7 > 0$ and $c_7 + c_8 > 0$ such that $v_*^i(\alpha^{-i}) = c_1\alpha^{-i} + c_2 + (c_3\alpha^{-i} + c_4)(c_5\alpha^{-i} + c_6)/(c_7\alpha^{-i} + c_8)$. The claim follows since

$$\frac{\partial^2 v_*^i(\alpha^{-i})}{\partial(\alpha^{-i})^2} = \frac{2(c_3c_8 - c_4c_7)(c_5c_8 - c_6c_7)}{(c_7\alpha^{-i} + c_8)^3}$$

has a constant sign because the denominator is non-zero for any $\alpha^{-i} \in [0, 1]$.

For any product set $\mathcal{A}_\diamond \subseteq \mathcal{A}(y)$, let $\mathcal{X}_{y,\mathcal{A}_\diamond}(\alpha)$ denote the set of all payoffs pairs v that can be written as $v = g(y,\alpha) + \delta_0\lambda(y,\alpha) + r\delta_0(y_s)$ for some action profile α supported on $\mathcal{A}_\diamond$ and $\delta_0 \in \Psi_{y,\mathcal{A}_\diamond}(\alpha)$. Let $\mathcal{X}_{y,\mathcal{A}_\diamond}$ denote the union over all such sets $\mathcal{X}_{y,\mathcal{A}_\diamond}(\alpha)$ over all α supported on $\mathcal{A}_\diamond$.

Claim 10. $\mathcal{X}_{y,\{1\}\times\mathcal{A}^2}(\alpha^2)$ is the set of all payoffs $(v^1, v_*^2(1))$ that satisfy $v^1 \geq v_*^1(\alpha^2)$. Consequently, $\mathcal{X}_{y,\{1\}\times\mathcal{A}^2}$ is the set of all payoffs $(v^1, v_*^2(1))$ that satisfy $v^1 \geq w^1$ for some $w \in \mathcal{X}_{y,\mathcal{A}}$. If player 1 plays action 0 instead, the same statements hold with reversed inequalities.

Fix some $v_0 \in \mathcal{X}_{y,\{1\}\times\mathcal{A}^2}(\alpha^2)$, decomposed by $(1, \alpha^2)$ and δ_0 enforcing $(1, \alpha^2)$. Then δ incentivizes player 1 to play action 1 against α^2 if and only if $\delta^1(y_s) \geq \delta_*^1(\alpha^2, y_s)$. Thus, such v_0 must satisfy $v_0^1 \geq v_*^1(\alpha^2)$. Since $v_*(1, \alpha^2) \in \mathcal{X}_{y,\mathcal{A}}$, the claim follows.

Fix now an arbitrary stationary payoff pair $w_0 \in \mathcal{S}_{y,\{a^1\}\times\mathcal{A}^2}(\mathcal{W})$ decomposed by $(1, \alpha^2)$ and $\delta_0 \in \Psi_{y,\{1\}\times\mathcal{A}^2}$ and abbreviate $v_0 := w_0 + r\delta_0(y_s)$. It follows from Lemma F.7 that $\delta_0^2 = \delta_*^2(1)$ and $\delta_0^1 \geq \delta_*^1(\alpha^2)$. If $\delta_0^1 = \delta_*^1(\alpha^2)$, then $w_0 = w_*(1, \alpha^2)$, hence $w_0 \in \mathcal{S}_{y,\mathcal{A}}(\mathcal{W})$. Suppose, therefore, that $\delta_0^1 > \delta_*^1(\alpha^2)$, which implies $v_0^1 > v_*^1(1, \alpha^2)$. We will show that w_0 lies in the desired convex hull by appealing to the following claim in various cases.

Claim 11. Suppose there exist $\alpha_1^2 \leq \alpha^2$, $\alpha_2^2 \geq \alpha^2$, and $\delta_k \in \Psi_{y,(1,\alpha_k^2)}$ for $k = 1, 2$ such that

$$g^1(y, 1, \alpha_k^2) + \delta_k^1\lambda(y, 1, \alpha_k^2) + r\delta_k^1(y_s) = v_0^1. \tag{39}$$

Then there exist $\eta \in [0, 1]$ and $w_k \in \mathcal{S}_{y,\text{supp}(1,\alpha_k^2)}(\mathcal{W})$ for $k = 1, 2$ with $w_0 = \eta w_1 + (1 - \eta)w_2$.

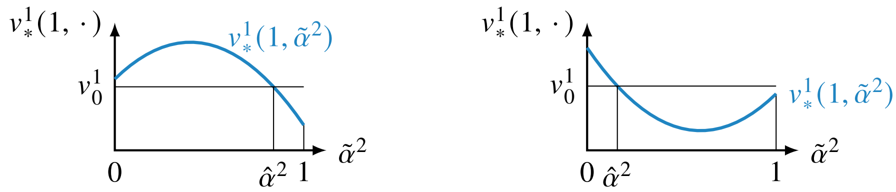

Figure 18: Illustration of monotonicity properties of $v^1_*(1,\tilde{\alpha}^2)$.

Without loss of generality we may assume that $\delta^2_k = \delta^2_*(1)$ since $\delta^2_*(1)$ makes player 2 indifferent between both pure actions. Set $w_k := g(y,1,\alpha^2_k) + \delta_k\lambda(y,1,\alpha^2_k)$ and $v_k := w_k + r\delta_k(y_s)$ and observe that $w^2_k = w^2_0$ and $v_k = v_0$. In particular, $v_k \in \mathcal{X}_{y,\mathrm{supp}(1,\alpha^2_k)} \cap \mathcal{W}_{y_s}$ and $w_k \in \mathcal{S}_{y,\mathrm{supp}(1,\alpha^2_k)}(\mathcal{W})$. Since α^2 is a convex combination of α^2_1 and α^2_2, taking convex combinations in (39) shows that $\delta^1_0(y_s)$ must lie between $\delta^1_1(y_s)$ and $\delta^1_2(y_s)$, hence w^1_0 lies between $w^1_k = v_0 - r\delta^1_k(y_s)$ for $k = 1,2$.

We are ready to show that $\mathcal{S}_{y,\{1\}\times\mathcal{A}^2}(\mathcal{W})$ is contained in the desired convex hull.

(i) Suppose first that v_0 lies in both $\mathcal{X}_{y,(1,1)}$ and $\mathcal{X}_{y,(1,0)}$, equivalent to $v^1_0 \geq v^1_*(1,a^2)$ for either pure action a^2. The statement thus follows from Claim 11 for $\alpha^2_1 = 0$, $\alpha^2_2 = 1$, and suitable $\delta_k \in \Psi_{y,(1,k-1)}$ for $k = 1,2$ that attain v_0, which must exist since $v_0 \in \mathcal{X}_{y,(1,k-1)}$.

(ii) Suppose next that v_0 lies in $\mathcal{X}_{y,(1,1)}$ but not in $\mathcal{X}_{y,(1,0)}$, equivalent to $v^1_*(1,1) \leq v^1_0 < v^1_*(1,0)$. Because $v^1_*(1,\tilde{\alpha}^2)$ is either convex or concave by Claim 9, there is a unique $\hat{\alpha}^2 \in (0,1)$ such that $v^1_*(1,\hat{\alpha}^2) = v^1_0$ and $v^1_*(1,\tilde{\alpha}^2) < v^1_0$ if and only if $\tilde{\alpha}^2 > \hat{\alpha}^2$; see Figure 18. In particular, $v^1_0 > v^1_*(1,\alpha^2)$ implies $\hat{\alpha}^2 < \alpha^2$, hence the statement now follows by applying Claim 11 to $\alpha^2_1 = \hat{\alpha}^2$, $\delta_1 = \delta_*(1,\hat{\alpha}^2)$, $\alpha^2_2 = 1$, and suitable $\delta_2 \in \Psi_{y,(1,1)}$ attaining $v_0 \in \mathcal{X}_{y,(1,1)}$.

(iii) The case $v_0 \in \mathcal{X}_{y,(1,0)} \setminus \mathcal{X}_{y,(1,1)}$ is completely analogous.

(iv) Finally, suppose that v_0 is neither in $\mathcal{X}_{y,(1,1)}$ nor in $\mathcal{X}_{y,(1,0)}$, equivalent to $v^1_0 < v^1_*(1,a^2)$ for either pure action a^2. Then $v^1_*(1,\tilde{\alpha}^2)$ must be strictly concave. Since $v^1_0 > v^1_*(1,\alpha^2)$, it cannot be that v^1_0 minimizes $v^1_*(1,\tilde{\alpha}^2)$, hence there exist $0 < \alpha^2_1 < \alpha^2 < \alpha^2_2 < 1$ such that $v^1_*(1,\alpha^2_k) = v^1_0$ for $k = 1,2$. The statement follows from Claim 11 for $\delta_k = \delta_*(1,\alpha^2_k)$.

For the converse, note that $\mathcal{X}_{y,\{a^1\}\times\mathcal{A}^2}(\alpha) \cap \mathcal{W}_{y_s}$ is continuous in α^2 by Claim 10, hence so is

$$\mathcal{S}_{y,\{a^1\}\times\mathcal{A}^2}(\alpha^2,\mathcal{W}) = \frac{r}{r+\lambda(y,a^1,\alpha^2)} g(y,a^1,\alpha^2) + \frac{r}{r+\lambda(y,a^1,\alpha^2)} \mathcal{X}_{y,\{a^1\}\times\mathcal{A}^2}(\alpha) \cap \mathcal{W}_{y_s}.$$

Fix any two $w_1, w_2 \in \mathcal{S}_{y,\mathcal{A}}(\mathcal{W}) \cup \mathcal{S}_{y,(a^1,1)}(\mathcal{W}) \cup \mathcal{S}_{y,(a^1,0)}(\mathcal{W})$ with $w^2_k = v^2_*(a^1)$ for $k = 1,2$.

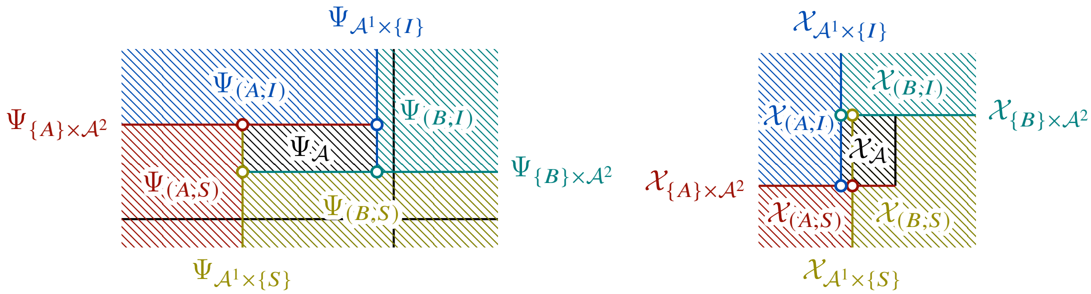

Figure 19: $\Psi_{\mathcal{A}_\diamond}$ and $\mathcal{X}_{\mathcal{A}_\diamond}$ in the regime-change game. The relationship between $\Psi_{\mathcal{A}}$ and $\mathcal{X}_{\mathcal{A}}$ is concave as evidenced by the fact that player 1's payoff in $\mathcal{X}_{\mathcal{A}}$ is maximized at a mixed action profile.

They must be decomposed by $(a^1, \alpha_k^2, \delta_k)$ with $\delta_k^2 = \delta_*^2(a^1)$, hence $w_k \in \mathcal{S}_{y,\{a^1\}\times\mathcal{A}^2}(\alpha_k^2, \mathcal{W})$. By continuity, any convex combination of w_1 and w_2 must lie in that set for some $\tilde{\alpha}^2$. □

Proof of Lemma 7.2. Fix a state y with unique successor state y_s. We can write $\partial\mathcal{K}_{y,\mathcal{A}_\diamond}(\mathcal{W})$ as the set of all payoffs w where the convex set $(\mathcal{W}_{y_s} - w)/r$ "leaves" $\Psi_{y,\mathcal{A}_\diamond}$, defined here as having a non-empty intersection at w and an empty intersection for some v arbitrarily close to w.

Since λ is assumed to be strictly monotone for both players, Lemma F.7 implies that δ_*^i is monotone in α^{-i} for each i, hence $\Psi_{y,\mathcal{A}}$ is a rectangle with extremal points $\delta_*(a)$ for all $a \in \mathcal{A}(y)$. Moreover, Lemma F.7 implies that the set of incentives for one-sided mixtures satisfies

$$\Psi_{y,\mathcal{A}^i\times\{1\}} = \left\{\delta \in \mathbb{R}^2 \;\middle|\; \delta^i = \delta_*^i(1) \text{ and } \delta^{-i} \geq \min\{\delta_*^1(0), \delta_*^1(1)\}\right\}$$

and similarly for $\Psi_{y,\mathcal{A}^i\times\{0\}}$; see Figure 19 for these sets in the regime-change game. In particular, incentives of one-sided mixtures are precisely the boundaries of $\Psi_{y,a}$ and $\Psi_{y,\mathcal{A}}$. Thus, if $(\mathcal{W}_{y_s} - w)/r$ leaves $\Psi_{y,a}$ or $\Psi_{y,\mathcal{A}}$, it must also leave one of the sets $\Psi_{y,\mathcal{A}^i\times\{a^{-i}\}}$.

For the converse inclusion, we need to show slightly more. Just because $(\mathcal{W}_{y_s} - w)/r$ leaves the boundary of $\mathcal{K}_{y,a}(\mathcal{W})$ or $\mathcal{K}_{y,\mathcal{A}}(\mathcal{W})$ need not imply that it leaves the entire set. We need to show that $\Psi_{y,\mathcal{A}^i\times\{a^{-i}\}}$ is "sandwiched" between the sets $\Psi_{y,a}$ and $\Psi_{y,\mathcal{A}}$ in the following sense.

Claim 12. If δ is extremal in $\Psi_{y,\mathcal{A}^i\times\{a^{-i}\}}$ with outward normal vector N, then δ is either extremal in $\Psi_{y,\mathcal{A}}$ or some $\Psi_{y,a}$ or with outward normal vector N.

For the sake of specificity, consider $\Psi_{y,\{1\}\times\mathcal{A}^2}$ of all δ with $\delta^2 = \delta_*^2(1)$ and $\delta^1 \geq \delta_*^1(\underline{a}^2)$, where $\underline{a}^2$ minimizes $\delta_*^1(\alpha^2)$. Fix any $\delta \in \Psi_{y,\{1\}\times\mathcal{A}^2}$ and suppose first that $\delta^1 > \delta_*^1(\underline{a}^2)$. Then there are only two outward normal vectors $N = \pm e_2$. Observe that $\Psi_{y,\{1\}\times\mathcal{A}^2}$ is precisely the upper bound of $\Psi_{y,(1,\underline{a}^2)}$ and the lower bound of $\Psi_{y,(1,\bar{a}^2)} \cup \Psi_{y,\mathcal{A}}$ or vice versa, depending on whether $\underline{a}^2$ is 1 or 0, hence the claim follows. If $\delta^1 = \delta^1_*(\underline{a}^2)$, then because $\Psi_{y,(1,\underline{a}^2)}$ and $\Psi_{y,\mathcal{A}}$ are both rectangles with one of them having its upper left corner and the other having its lower left corner at δ, the claim follows.

Fix now an arbitrary boundary point w of $\mathcal{K}_{y,\{1\}\times\mathcal{A}^2}(\mathcal{W})$. Then the convex set $(\mathcal{W}_{y_s} - w)/r$ must touch $\Psi_{y,\{1\}\times\mathcal{A}^2}$ at some δ, i.e., there exists an outward normal vector N to $\Psi_{y,\{1\}\times\mathcal{A}^2}$ such that the two convex sets $\Psi_{y,\{1\}\times\mathcal{A}^2}$ and $(\mathcal{W}_{y_s} - w)/r$ are separated by the line orthogonal to N. In particular, $(\mathcal{W}_{y_s} - w + \varepsilon N)/r$ is strictly separated from $\Psi_{y,\{1\}\times\mathcal{A}^2}$ for any $\varepsilon > 0$. It follows from Claim 12 that one of the sets $\Psi_{y,(1,\underline{a}^i)}$, $\Psi_{y,(1,\bar{a}^i)}$, and $\Psi_{y,\mathcal{A}}$ also intersects $(\mathcal{W}_{y_s} - w)/r$ and is strictly separated from $(\mathcal{W}_{y_s} - w + \varepsilon N)/r$ for any $\varepsilon > 0$. □

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