Online Appendix to "Bail-ins and Bail-outs: Incentives, Connectivity, and Systemic Stability"

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This online appendix contains the characterization of the equilibrium asset recovery rate and provides an algorithm for computing the equilibrium bail-in of Theorem 3.8. It also contains proofs for the results in Sections 4 and 5, as well as some auxiliary results omitted from the main text.

D Characterization of Equilibrium Asset Recovery Rate

This appendix provides more details on the results presented in Section 3, characterizing the equilibrium asset recovery rate α_* in Theorem 3.8 as well as the equilibrium amount of welfare burnt. We also present an iterative algorithm to compute $\mathcal{C}_*$ and α_*.

As in the main body, we begin by considering bail-ins where each bank i in some set $\mathcal{C}$ contributes $\eta^i(\alpha, \ell)$ and each bank $i \notin \mathcal{C}$ receives subsidies $s^i(\alpha, \ell)$ for a liquidation decision ℓ that induces recovery rate α, where $\eta(\alpha, \ell)$ and $s(\alpha, \ell)$ are defined in (13) and (17), respectively. For ease of reference, the set of all such bail-ins is formalized in the following definition.

Definition D.1. Let $\Xi_*(\mathcal{C}, \alpha)$ denote the set of all bail-ins (b, s) satisfying

(i) $\underline{b}^i \leq b^i - s^i \leq b^i_*(\alpha)$ for every $i \in \mathcal{C}$,

(ii) $b^i = 0$ and $s^i(\alpha, e) \leq s^i \leq s^i(\alpha, 0)$ for every $i \notin \mathcal{C}$,

(iii) $\sum_{i=1}^n \big(s^i(\alpha, 0) - s^i\big) + \sum_{i \in \mathcal{C}} \big(b^i - \underline{b}^i\big) = -\frac{\alpha \ln(\alpha)}{\gamma}$.

As stated by Lemma 3.7, welfare losses are minimized by bail-ins in $\Xi_*(\mathcal{C}, \alpha)$ among individually incentive-compatible bail-ins with contributing banks in $\mathcal{C}$. Similarly to (34), it follows that $\min(b, b_0) = \underline{b}$ for any $(b, s) \in \Xi_*(\mathcal{C}, \alpha)$. Therefore, Lemma 3.6 shows that the welfare impact of the banks' contributions is equal to

$$h_{\mathcal{C}}(\alpha) := g(\alpha_P) - g(\alpha) + \lambda \sum_{i \in \mathcal{C}} \underline{b}^i. \tag{45}$$

The welfare impact of contributions by banks in $\mathcal{C}$ depends on the liquidation decision only through the induced asset recovery rate. Thus, $h_{\mathcal{C}}$ is strictly concave and maximized at α_{ind}.

The regulator would like to induce recovery rate α_{ind} if it is possible to do so in equilibrium. It may not be possible if banks do not liquidate a sufficient quantity of assets to drive the recovery rate down to α_{ind} or if the no-free-riding constraint fails to hold for the resulting bail-in. The following lemma characterizes the lowest asset recovery rate $\alpha_{\mathcal{C}}$ that can be attained by any bail-in with contributing banks $\mathcal{C}$.

Lemma D.1. *For any set of banks $\mathcal{C}$, the equation*

$$\alpha = \exp\Big(-\gamma \sum_{i=1}^{n} \frac{1}{\alpha}\big(L^i + w^i + b_*^i(\alpha)1_{\{i\in\mathcal{C}\}} - c^i - s^i(\alpha,e) - (\pi L)^i\big)^+\Big) \tag{46}$$

admits at least one fixed point. Let $\alpha_{\mathcal{C}}$ denote the largest fixed point of (46). Then $\bar{\alpha}(b,s,1) \geq \alpha_{\mathcal{C}}$ for any complete, feasible, and individually incentive-compatible bail-in with contributing banks in $\mathcal{C}$. Moreover, $\alpha_{\mathcal{C}} \geq \frac{1}{\mathrm{e}}$ and $\Xi_*(\mathcal{C},\alpha) \neq \emptyset$ if and only if $\alpha \in [\alpha_{\mathcal{C}}, 1]$.

Since $\alpha_{\text{ind}} \leq 1$, Lemma D.1 shows that, without the no-free-riding constraints, the regulator would optimally attain asset recovery rate $\alpha^*_{\mathcal{C}} := \max(\alpha_{\text{ind}}, \alpha_{\mathcal{C}})$ for some set of contributing banks $\mathcal{C}$. The characterization in the following Theorem takes into account the free-riding constraints.

Theorem D.2. *For any integer k let $\mathcal{C}_k$ denote the set $\mathcal{C}$ of size k that maximizes $h_{\mathcal{C}}(\alpha^*_{\mathcal{C}})$. Set $W_k := W_P - h_{\mathcal{C}_k}(\alpha^*_{\mathcal{C}_k})$ and $m := \min(k \mid W_k < W_N)$. The function $\hat{W}_{\mathcal{C}}(\alpha) := W_P - h_{\mathcal{C}}(\alpha) + \hat{\chi}_{\mathcal{C}}(\alpha)$ is strictly quasi-concave for any set of banks $\mathcal{C}$. Let $\hat{\alpha}_{\mathcal{C}}$ denote the unique minimizer of $\hat{W}_{\mathcal{C}}$. For any bank $i \in \mathcal{C}$, denote $\tilde{\alpha}^i_{\mathcal{C}} := \sup\{\alpha \in [\alpha_{\mathcal{C}}, 1] \mid \chi^i_{\mathcal{C}}(\alpha) > 0\}$ and set $\tilde{\alpha}_{\mathcal{C}} := \alpha^*_{\mathcal{C}} \vee \max_{i\in\mathcal{C}} \tilde{\alpha}^i_{\mathcal{C}}$. Set*

$$\ell^i_* := \max\Big(\min\Big(L^i + w^i - c^i - \pi p^i_N + (1-\alpha_N)\ell^i_N, e^i\Big), 0\Big)$$

for any bank $i \in \mathcal{C}$ and set $\ell^i_* = 0$ for every $i \notin \mathcal{C}$. Define

$$\alpha(\mathcal{C}) := \begin{cases} \tilde{\alpha}_{\mathcal{C}} & \textit{if } \hat{\chi}_{\mathcal{C}}(\tilde{\alpha}_{\mathcal{C}}) = 0, \\ \max(\alpha(\ell_*), \hat{\alpha}_{\mathcal{C}}) & \textit{otherwise.} \end{cases}$$

The following statements hold for any set of banks $\mathcal{C}$:

(i) $W_{\mathcal{C}}$ is minimized at $\alpha(\mathcal{C})$.

(ii) $\alpha(\mathcal{C}) = \tilde{\alpha}_{\mathcal{C}} = \alpha^*_{\mathcal{C}}$ and $\chi_{\mathcal{C}}(\alpha^*_{\mathcal{C}}) = 0$ for any $\mathcal{C}$ with $|\mathcal{C}| \leq m$.

Moreover, the set of contributing banks in any subgame-Pareto efficient equilibrium is generically unique up to banks i with $b^i_0 = 0$. Any equilibrium bail-in with contributing banks in $\mathcal{C}$ induces asset recovery rate $\alpha(\mathcal{C})$.

For sufficiently small consortia of contributing banks, there is no incentive to free-ride and welfare is maximized at the unconstrained optimum $\alpha^*_{\mathcal{C}}$. For larger consortia, welfare is maximized at the asset recovery rate $\tilde{\alpha}_{\mathcal{C}}$ closest to the unconstrained optimum $\alpha^*_{\mathcal{C}}$, at which no welfare burning is necessary *if* the total liquidation implied by the no free-riding constraint is consistent with asset recovery rate $\tilde{\alpha}_{\mathcal{C}}$, that is, it does not depress the recovery rate below $\tilde{\alpha}_{\mathcal{C}}$.[1] If the total liquidation implied by the no free-riding constraint is inconsistent with the asset recovery rate $\tilde{\alpha}_{\mathcal{C}}$, the regulator requests lower contributions from banks which induce recovery rate $\max(\alpha(\ell_*), \hat{\alpha}_{\mathcal{C}})$ but do require welfare burning. At recovery rate $\max(\alpha(\ell_*), \hat{\alpha}_{\mathcal{C}})$, the consistency requirement binds. This implies the following corollary.

[1]At recovery rate α, the no free-riding condition imposes that a contributing bank $i \in \mathcal{C}$ liquidates the smallest amount $x^i \geq 0$ for which $\lambda b^i + g_\alpha(x^i) \geq W_N - W_P - h_{\mathcal{C}}(\alpha)$; see Lemma A.1. This liquidation amount is inconsistent with recovery rate α if and only if $\sum_{i\in\mathcal{C}} x^i > -\ln(\alpha)/\gamma$ because then, the required liquidation would depress the recovery rate below α by (1).

Corollary D.3. *Let $\mathcal{C}_*$ and α_* be as in Theorem D.2. Then $\chi_{\mathcal{C}_*}(\alpha_*) = \hat{\chi}_{\mathcal{C}_*}(\alpha_*)$.*

Theorem D.2 implies an algorithm that can be used to find the optimal set $\mathcal{C}_*$ of contributing banks. It suggests that the regulator performs an iteration over the size of the consortia of contributing banks, similarly as in Theorem 5.4. Different from the setting of Section 5, if asset buyers are inefficient, there is no global order according to which the regulator includes banks into the bail-in consortium. Rather, the set of most exposed banks changes as the regulator includes more banks into the consortium because the size of their contributions may impact their liquidation decisions, and thus affect the asset recovery rate. The set of liquidation decisions ℓ_k, the asset recovery rate α_k, and the set $\mathcal{C}_k$ of contributing banks for each k is a fixed point of (46) such that ℓ_k induces asset recovery rate α_k, contributions $\eta^i(\alpha_k, \ell_k)$ are maximized by banks in $\mathcal{C}_k$, and the contributions $\eta^i(\alpha_k, \ell_k)$ by banks in $\mathcal{C}_k$ at asset recovery rate α_k induce the liquidation vector ℓ_k. Nevertheless, since $\eta^i(\alpha, \ell)$ is piecewise linear in α, the set of banks most exposed to contagion does not change too frequently and one can use the set $\mathcal{C}'_{k+1}$ of $k+1$ most-exposed banks using (α_k, ℓ_k) as a starting point for constructing the set $\mathcal{C}_{k+1}$. If there exists ℓ' with $\alpha(\ell') = \alpha(\mathcal{C}'_{k+1})$ such that $\mathcal{C}'_{k+1}$ is the set of $k+1$ most exposed banks for $(\alpha(\ell'), \ell')$, then $\mathcal{C}_{k+1} = \mathcal{C}'_{k+1}$. Different from Theorem 5.4, the optimization does not stop after including $m+1$ banks: adding more banks into the consortium can increase welfare even after adding $m+1$ banks if the contributions by these additional banks induce a smaller amount of asset liquidation. As a function of k, the asset recovery rate is decreasing until m and increasing thereafter.

E Existence, Monotonicity, and Differentiability of Asset Recovery Rates

This section contains results related to the asset recovery rate, which are omitted from the main text. We show that α_p, for a repayment vector p, is differentiable. We also show existence of the fixed point $\alpha_{\mathcal{C}}$ defined in Lemma D.1. Lastly, we provide an elementary construction of the asset recovery rate which allows us to conclude that $\bar{\alpha}(b, s, 1) \geq \frac{1}{\mathrm{e}}$ for complete rescues (b, s) in the proof of Lemma C.2.

Lemma E.1. *Let α_p be defined as in Lemma 2.2. The recovery rate α_p is continuous in L, π, c, w, e, and p. Moreover, it is differentiable in these variables where $\mathcal{D} = \{i \mid L^i + w^i > c^i + \alpha_p e^i + (\pi p)^i\}$ and $\mathcal{I} = \{i \notin \mathcal{D} \mid L^i + w^i > c^i + (\pi p)^i\}$ are constant.*

In the proof of Lemma E.1, we need the following auxiliary result, Lemma E.2, which is also invoked in the proof of Lemma C.2 in the main text.

Lemma E.2. *Let $x, y \geq 0$ parametrize the function $f_{x,y}(\alpha) = \exp\left(-\frac{x}{\alpha} - y\right)$ and let $x_0 = \exp(-y-1)$. The function $f_{x,y}$ has a fixed point on $(0, 1]$ if and only if $x \leq x_0$. The fixed point is unique at $\alpha = \exp(-y)$ if $x = 0$, it is unique at $\alpha = x_0$ if $x = x_0$, and otherwise there are two fixed points, one on $(0, x_0)$ and one on $(x_0, 1)$. The fixed point on $(0, x_0)$ is increasing in x, the fixed point on $(x_0, 1)$ is decreasing in x, and both are differentiable in x.*

Proof. Fix $y \geq 0$. The statement is obvious if $x = 0$. We thus restrict our attention to the case $x > 0$. Then $f_{x,y}(1) < 1$ and $\lim_{\alpha \to 0+} f_{x,y}(\alpha) = 0$. Since $f_{x,y}$ is continuously differentiable on $(0, 1]$ with $\lim_{\alpha \to 0+} f'_{x,y}(\alpha) = 0$, this implies that the graph of $f_{x,y}$ lies below the identity graph immediately to the right of 0 and immediately to the left of 1. Observe next that

$$f''_{x,y}(\alpha) = f'_{x,y}(\alpha)\frac{x}{\alpha^2} - f_{x,y}(\alpha)\frac{2x}{\alpha^3} = \frac{f_{x,y}(\alpha)}{\alpha^4}(x^2 - 2x\alpha).$$

Therefore, $f_{x,y}$ is convex for sufficiently large x, which implies that the graph of $f_{x,y}$ is below the identity graph on the entire interval. In particular, $f_{x,y}$ has no fixed points for sufficiently large x.

Because $f_{x,y}$ is differentiable for any x and the graph of $f_{x,y}$ changes continuously in x, the number of fixed points can change only at a point x_* where the graph of $f_{x_*,y}$ and the identity graph touch, that is, where $f'_{x_*,y}(\alpha_0) = 1$ for a fixed point α_0 of $f_{x_*,y}$. Moreover, for each such touching point α_0, the number of fixed points can increase or decrease by at most 2.

Suppose that there exists x_* with "touching point" α_0. Then $1 = f'_{x_*,y}(\alpha_0) = f_{x_*,y}(\alpha_0)\frac{x_*}{\alpha_0^2} = \frac{x_*}{\alpha_0}$ shows that $\alpha_0 = x_*$, i.e., each such x_* admits only one touching point. Moreover, it follows that $x_* = f_{x_*,y}(x_*) = \exp(-y-1) = x_0$, i.e., such a point x_* is unique and coincides with x_0. Therefore, the number of fixed points of $f_{x,y}$ can change only at $x = x_0$. Since $f_{x,y}$ has zero fixed points for sufficiently large x, it follows that $f_{x,y}$ has no fixed points for $x > x_0$. Moreover, because there is only one touching point at x_0, $f_{x,y}$ can have at most two fixed points for $x < x_0$. Since $f_{x,y}(\alpha)$ is decreasing in x for each α, it follows that $f_{x,y}(x_0) > f_{x_0,y}(x_0) = x_0$, i.e., the graph of $f_{x,y}$ at x_0 is above the identity graph. Since the graph of $f_{x,y}$ is below the identity graph at 1 and immediately to the right of 0, this shows that $f_{x,y}$ has exactly two fixed points for $x \in (0, x_0)$, a fixed point $\alpha_1(x)$ in the interval $(0, x_0)$ and a fixed point $\alpha_2(x)$ on $(x_0, 1)$.

Finally, since $f_{x,y}$ is continuously decreasing in x for each α, it follows that $\alpha_1(x)$ is continuously increasing and $\alpha_2(x)$ is continuously decreasing in x. Continuity and monotonicity imply that $\alpha_1(x)$ and $\alpha_2(x)$ are differentiable almost everywhere. Taking the derivative of $\alpha_i(x) = \exp(-\frac{x}{\alpha_i(x)} - y)$ at a differentiability point yields

$$\alpha'_i(x) = \alpha_i(x)\left(-\frac{1}{\alpha_i(x)} + \frac{x}{\alpha_i^2(x)}\alpha'_i(x)\right) = \frac{\alpha_i(x)}{x - \alpha_i(x)},$$

where we have used that $\alpha_1(x) < x < \alpha_2(x)$. Since α_i is continuous in x, this shows that its derivative is continuous as well and, therefore, α_i is differentiable everywhere on $(0, x_0)$. □

Proof of Lemma E.1. Fix a vector p. Note that $\ell(p, \alpha_p) = \min(\frac{1}{\alpha_p}(L + w - c - \pi p)^+, e)$ and hence

$$\alpha_p = \exp\left(-\frac{\gamma}{\alpha_p}\sum_{i \in \mathcal{I}}(L^i + w^i - c^i - (\pi p)^i) - \gamma\sum_{i \in \mathcal{D}} e^i\right). \tag{47}$$

Lemma 2.2 shows that α_p is the largest fixed point of (47). In particular, it is the largest fixed point of the function $f_{x,y}(\alpha)$ in Lemma E.2 for $y = \gamma\sum_{i \in \mathcal{D}} e^i$ and $x = \gamma\sum_{i \in \mathcal{I}}(L^i + w^i - c^i - (\pi p)^i)$. Lemma E.2 thus implies that α_p is differentiable in x and y, and hence also in γ, L, c, w, e, and π where $\mathcal{D}$ and $\mathcal{I}$ are constant. Continuity follows because $\ell(p, \alpha_p)$ is continuous. □

The construction of the clearing equilibrium as an iterated fixed point of monotone operators in the proof of Lemma 2.1 yields the following monotonicity properties of clearing equilibria.

Lemma E.3. *Consider a bail-in (b, s) in the financial system $(L, \pi, w, c, e, \beta, \gamma)$.*

1. *For a fixed clearing payment vector p, the asset recovery rate α_p is non-decreasing in c^j, s^j, and p^j for any j and non-increasing in γ, L^j, w^j, b^j, and e^j for any j.*

2. *For a fixed clearing payment vector p, for each bank i, ℓ_p^i is non-increasing in c^j, s^j, and p^j for any j and non-decreasing in γ, L^j, w^j, b^j, and e^j for any j*

3. *Let $(\bar{p}, \bar{\ell}, \bar{\alpha})$ be the clearing equilibrium of the financial system. Then $\bar{\alpha}$ and $\bar{p}^i$ for any i are non-decreasing in β, c^j, and s^j and non-increasing in γ, w^j, and b^j. Moreover, $\bar{\ell}^i$ for any i is non-increasing in β, c^j, and s^j and non-decreasing in γ, w^j, and b^j.*

Proof. Fix p and let Φ_p be defined as in the proof of Lemma 2.2. Note that

$$\Phi_p^i(x) = \min\left(\exp\left(\gamma \sum_{i=1}^n x^i\right)\left(L^i + w^i + b^i - c^i - s^i - (\pi p)^i\right)^+, e^i\right)$$

is non-increasing in c^j, s^j, and p^j for any j and non-decreasing in γ, L^j, w^j, b^j, and e^j for any j. The second statement thus follows from a similar application of Lemma B.3 as in the proof of Lemma 2.1. Since α_p is a decreasing function of $\sum_{i=1}^n \ell_p^i$, it follows immediately that α_p is non-decreasing in c^j, s^j, and p^j for any j and non-increasing in L^j, w^j, b^j, and e^j for any j. To show monotonicty of α_p in γ, observe that α_p is differentiable almost everywhere by Lemma E.1. Taking the weak partial derivative in (1) thus yields

$$\frac{\partial \alpha_p}{\partial \gamma} = -\alpha_p \left(\sum_{i=1}^n \ell_p^i + \gamma \sum_{i=1}^n \frac{\partial \ell_p^i}{\partial \gamma}\right) \leq 0. \tag{48}$$

This concludes the proof of the first statement. Let now Φ be defined as in (29). Since α_p is non-decreasing in β, c^j and s^j for any j and non-increasing in γ, w^j and b^j for any j, it follows immediately that so is $\Phi^i(p)$. Another application of Lemma B.3 thus shows that $\bar{p}$ is non-decreasing in β, c^j and s^j for any j and non-increasing in γ, w^j and b^j for any j. Therefore, $\bar{\alpha} = \alpha_{\bar{p}}$ is non-decreasing in β, c^j and s^j for any j and non-increasing in γ, w^j and b^j for any j. Since $\bar{\ell} = \ell(\bar{p}, \bar{\alpha})$ is a non-increasing function of $\bar{p}$ and $\bar{\alpha}$, the last statement follows. □

Lemma E.2 also helps us establish the existence of the asset recovery rate $\alpha_{\mathcal{C}}$. In the following lemma we establish existence, but we defer the remaining statements of Lemma D.1 to a later point.

Lemma E.4. *Let $v_{\mathcal{C}}(\alpha)$ denote the right-hand side of* (46)*. For any set of banks $\mathcal{C}$,* (46) *has at least one fixed point. Let $\alpha_{\mathcal{C}}$ denote the largest fixed point. For $\alpha \geq \frac{1}{\mathrm{e}}$, there is equivalence between $v_{\mathcal{C}}(\alpha) \leq \alpha$ and $\alpha \geq \alpha_{\mathcal{C}}$.*

Proof of Lemma E.4. Fix a set of banks $\mathcal{C}$. We show the existence of a fixed point using a similar argument as in the proof of Lemma 2.2. Define the vector $\varphi(\alpha)$ by setting

$$\varphi^i(\alpha) := \frac{1}{\alpha}\big(L^i + w^i + b_*^i(\alpha)1_{\{i\in\mathcal{C}\}} - c^i - s^i(\alpha, e) - (\pi L)^i\big)^+.$$

It follows from the definitions of $s^i(\alpha, e)$ and $b_*^i(\alpha)$ in (17) and (14), respectively, that φ^i is non-increasing in α and that $\varphi^i(\alpha) \in [0, e^i]$ for every bank i. Thus, $\alpha \circ \varphi$ is a non-decreasing operator that maps $\mathcal{L} := \bigtimes_{i=1}^n [0, e^i]$ into itself, where the function α is defined as in (1). By Tarski's fixed point theorem, there exists a lowest fixed point $\underline{\ell}$ of $\alpha \circ \varphi$. The greatest fixed point of $v_{\mathcal{C}}$ is thus given by $\alpha_{\mathcal{C}} := \alpha(\underline{\ell})$. For the second statement, observe that the definitions of $\underline{b}$ and $s(\alpha, 0)$ imply that $v_{\mathcal{C}}(\alpha) = f_{x,y}(\alpha)$ for suitable non-negative x and y. Since $v_{\mathcal{C}}(\alpha) = f_{x,y}(\alpha)$ admits a fixed point, it follows from Lemma E.2 that $\alpha_{\mathcal{C}}$ is the unique fixed point on $\left[\frac{1}{\mathrm{e}}, 1\right]$. The graph of the function $f_{x,y}$ thus intersects the identity graph precisely once on the interval $\left[\frac{1}{\mathrm{e}}, 1\right]$. Because $f_{x,y}(1) \le 1$, it follows that $v_{\mathcal{C}}(\alpha) \le \alpha$ on that interval if and only if $\alpha \ge \alpha_{\mathcal{C}}$. □

F Proof of Theorem D.2

In the proof of Theorem D.2, we crucially rely on bail-in proposals from the set $\Xi_*(\mathcal{C}, \alpha)$. Lemma 3.7 in the main text shows that a complete, feasible, and individually incentive-compatible bail-in (b, s) attains the lower bound on welfare losses in (15) only if $(b, s) \in \Xi_*(\mathcal{C}, \alpha)$. In the following result, we show that the converse is true as well.

Lemma F.1. *For any set of banks $\mathcal{C}$ and any $\alpha \ge \frac{1}{\mathrm{e}}$, any $(b, s) \in \Xi_*(\mathcal{C}, \alpha)$ is a complete, feasible, and individually incentive-compatible bail-in with $\bar{\alpha}(b, s, 1) = \alpha$ and*

$$W_\lambda(b, s, 1) = W_P - h_{\mathcal{C}}(\alpha).$$

The condition $\alpha \ge \frac{1}{\mathrm{e}}$ is necessary to ensure that the induced recovery rate indeed coincides with α: Condition (iii) in the Definition D.1 states that the total shortfall in the bail-in $(b, s) \in \Xi_*(\mathcal{C}, \alpha)$ is equal to $-\alpha \ln(\alpha)/\gamma$. This pins down the asset recovery rate uniquely because the asset recovery rate is given by the largest solution to the equation in Condition (iii). However, since $-\alpha\ln(\alpha)/\gamma = x$ has two solutions in general, the attained asset recovery rate is equal to α only if $\alpha \ge \frac{1}{\mathrm{e}}$ as stated by the following lemma. This places no restrictions on attainability of complete bail-ins because any such bail-in induces a recovery rate of at least $\frac{1}{\mathrm{e}}$ by Lemma C.2.

Lemma F.2. *The function $z(\alpha) = \alpha\ln(\alpha)$ is strictly convex on $[0, 1]$, strictly decreasing on $\left[0, \frac{1}{\mathrm{e}}\right]$, strictly increasing on $\left[\frac{1}{\mathrm{e}}, 1\right]$, and $z\left(\frac{1}{\mathrm{e}}\right) = -\frac{1}{\mathrm{e}}$.*

Proof. This follows directly from the the first two derivatives $z'(\alpha) = 1 + \ln(\alpha)$ and $z''(\alpha) = \frac{1}{\alpha}$. □

Proof of Lemma F.1. Fix $\mathcal{C}$, $\alpha \ge \frac{1}{\mathrm{e}}$ and a bail-in $(b, s) \in \Xi_*(\mathcal{C}, \alpha)$. By Lemma C.1 we may assume without loss of generality that $s^i = 0$ for $i \in \mathcal{C}$. Define the vector ℓ by setting $\ell^i = \frac{1}{\alpha}(b^i - \underline{b}^i)$ for

$i \in \mathcal{C}$ and $\ell^i = \frac{1}{\alpha}(s^i(\alpha, 0) - s^i)$ for $i \notin \mathcal{C}$. We will show that (L, ℓ, α) is a solution to (1), (2), and (3) and hence, it must be the clearing equilibrium of the financial system after transfers (b, s). It follows from the definition of ℓ that $b^i = \eta^i(\alpha, \ell)$ for $i \in \mathcal{C}$ and $s^i = s^i(\alpha, \ell)$ for $i \notin \mathcal{C}$. The definitions of $\eta(\alpha, \ell)$ and $s(\alpha, \ell)$ in (13) and (17) imply that for every bank i,

$$b^i - s^i \le c^i + \alpha \ell^i + (\pi L)^i - L^i - w^i. \tag{49}$$

The inequality (49) shows that (3) is satisfied, i.e., every bank is solvent under the clearing payment vector L. For banks $i \notin \mathcal{C}$, it follows immediately from the definition of ℓ^i and $s^i(\alpha, e)$ that (2) is satisfied for clearing vector L and asset recovery rate α. The same is true for $i \in \mathcal{C}$ if $\underline{b}^i = b_0^i$. If $\underline{b}^i < b_0^i$ for $i \in \mathcal{C}$ instead, then $b_*^i(\alpha) = \underline{b}^i$, which implies that $b^i = \underline{b}^i < b_0^i$. In particular, $\ell^i = 0$ satisfies (2) for any p, α. Finally, using the definition of ℓ together with Condition (iii) in Definition D.1, it is easily seen that (1) is satisfied. Since $\alpha \ge \frac{1}{\mathrm{e}}$, Lemma F.2 shows that α is the largest asset recovery rate satisfying (1), hence (L, ℓ, α) is the Pareto efficient clearing equilibrium after transfers (b, s). This shows that (b, s) is a complete, feasible bail-in proposal with asset recovery rate $\bar{\alpha}(b, s, 1) = \alpha$. Moreover, (b, s) is individually incentive-compatible since $b^i \le b_*^i(\alpha)$. It now follows from Lemma 3.7 and the definition of $h_{\mathcal{C}}$ that welfare losses are of the desired form. □

Proof of Lemma D.1. Existence of the fixed point has been established in Lemma E.4. Let (b, s) be any complete, feasible, and individually incentive-compatible rescue with contributing banks $\mathcal{C}$. It follows from (33) that any bank i liquidates an amount

$$\ell^i(b, s, 1) = \frac{1}{\alpha}\left((s_0^i - s^i)^+ + (b^i - b_0^i)^+\right). \tag{50}$$

Since every bank is rescued in (b, s), any bank $i \notin \mathcal{C}$ must receive subsidies of at least $s(\alpha, e)$. Therefore, (50) shows that $\alpha \ell^i = (s_0^i - s^i)^+ \le \left(s^i(\alpha, 0) - s^i(\alpha, e)\right)^+ = s^i(\alpha, 0) - s^i(\alpha, e)$. For $i \in \mathcal{C}$, we may assume that $s^i = 0$ by Lemma C.1. Individual incentive-compatibility implies via Condition 2 of Lemma 3.5 that $b^i \le b_*^i(\alpha)$. For b_0^i defined as in Lemma 3.6, we obtain

$$\alpha \ell^i = (b^i - b_0^i)^+ \le \left(b_*^i(\alpha) - b_0^i\right)^+ = b_*^i(\alpha) - \underline{b}^i. \tag{51}$$

It now follows that $\alpha = \exp(-\gamma \sum_{i=1}^n \ell^i) \ge v_{\mathcal{C}}(\alpha)$, where the latter function is defined in Lemma E.4. Because $\alpha \ge \frac{1}{\mathrm{e}}$ by Lemma C.2, Lemma E.4 implies that $\alpha \ge \alpha_{\mathcal{C}}$. Together with Lemma F.1, this implies that $\Xi_*(\mathcal{C}, \alpha) = \emptyset$ if $\alpha < \alpha_{\mathcal{C}}$. □

Our next result shows that welfare burning is not needed for sufficiently small rescue consortia because there is no incentive to free-ride.

Lemma F.3. *Let m be defined as in Theorem D.2. For any set of banks $\mathcal{C}$ with $|\mathcal{C}| \le m$ and any $\alpha \ge \alpha_{\mathcal{C}}$, we have $\chi_{\mathcal{C}}(\alpha) = 0$.*

In the proof, we will rely on the following auxiliary lemma.

Lemma F.4. *Let (b, s) be a complete, feasible, and individually incentive-compatible bail-in with contributing banks $\mathcal{C}$. Let $k = |\mathcal{C}|$ and let W_k be defined as in Theorem D.2. Then $W_\lambda(b, s, 1) \geq W_k$ and $W_\lambda(b, s, 1) = W_k$ if and only if $(b, s) \in \Xi_*(\mathcal{C}_k, \alpha^*_{\mathcal{C}_k})$.*

Proof. Fix such a bail-in proposal (b, s) with k contributing banks. Lemma D.1 shows that $\alpha_{\mathcal{C}}$ is the smallest recovery rate that can be attained in any complete bail-in with contributing banks in $\mathcal{C}$. The maximum of $h_{\mathcal{C}}(\alpha)$ on $[\alpha_{\mathcal{C}}, 1]$ is thus attained at $\alpha^*_{\mathcal{C}}$. It follows that $h_{\mathcal{C}}(\alpha) \leq h_{\mathcal{C}}(\alpha^*_{\mathcal{C}}) \leq h_{\mathcal{C}_k}(\alpha^*_{\mathcal{C}_k})$ by definition of $\mathcal{C}_k$. Together with Lemma 3.7, this implies that

$$W_\lambda(b, s, 1) \geq W_P - h_{\mathcal{C}}(\alpha) \geq W_k. \tag{52}$$

The proof is concluded by observing that the first inequality in (52) holds with equality if and only if $(b, s) \in \Xi_*(\mathcal{C}, \alpha)$ and the second inequality (52) binds if and only if $\mathcal{C} = \mathcal{C}_k$ and $\alpha = \alpha^*_{\mathcal{C}_k}$. □

Proof of Lemma F.3. Fix a set of banks $\mathcal{C}$ with $|\mathcal{C}| \leq m$, an asset recovery rate $\alpha \geq \alpha_{\mathcal{C}}$, as well as a bail-in $(b, s) \in \Xi_*(\mathcal{C}, \alpha)$. By Lemma F.1, (b, s) is a complete, feasible, and individually incentive-compatible bail-in proposal. For any $i \in \mathcal{C}$, let a_{-i} denote the response vector, where every bank but bank i accepts the proposal. Lemmas 3.6 and F.4 imply that

$$W_\lambda(b, s, a_{-i}) = W_P + \big(g(\bar{\alpha}(b, s, a_{-i})) - g(\alpha_P)\big) - \lambda \sum_{j \in \mathcal{C} \setminus \{i\}} \eta^j(\alpha, 0) \geq W_{k-1} \geq W_N,$$

where the last inequality follows from the definition of m. This shows that no welfare burning is necessary for Condition 1 of Lemma 3.5 to hold. □

Our next result shows that m is well defined, i.e., the sequence $(W_k)_{k \geq 1}$ will decrease below W_N eventually. This result is invoked in the proof of Theorem 3.8.

Lemma F.5. *Let W_k be defined as in Theorem D.2. There exists k with $W_k < W_N$. In particular, m in Theorem D.2 is well defined and $(b, s) \in \Xi_*(\mathcal{C}_m, \alpha^*_{\mathcal{C}_m})$ admits accepting equilibrium $(1, \ldots, 1)$.*

Proof. We first show that there is a $k \leq n$ such that $W_k < W_N$. Let $\mathcal{C}_0 := \{1, \ldots, n\}$ denote the set of all banks, let $\mathcal{C} := \{i \mid b^i_*(\alpha_{\mathcal{C}_0}) > 0\}$ denote the largest possible set of contributing banks, and let $k := |\mathcal{C}|$. It follows from the definition of $\mathcal{C}$ that the smallest recovery rate that can be sustained in a complete bail-in is $\alpha_{\mathcal{C}} = \alpha_{\mathcal{C}_0}$. If $W_{k-1} < W_N$, then the statement holds trivially. Suppose, therefore, that $W_{k'} \geq W_N$ for every $k' < k$. We will show that then a bail-in with contributing banks $\mathcal{C}$ must be incentive compatible and attain welfare losses below W_N. Consider first the case $\alpha_N < \alpha_{\mathcal{C}}$. Define the bail-in (b, s) by setting $b^i = b^i_*(\alpha_{\mathcal{C}})$ and $s^i = s^i(\alpha_{\mathcal{C}}, e)$ for every bank i. It follows from Lemma F.1 that $(b, s) \in \Xi_*(\mathcal{C}, \alpha_{\mathcal{C}})$ is a complete, feasible, and individually incentive-compatible bail-in proposal with $\bar{\alpha}(b, s, 1) = \alpha_{\mathcal{C}}$. Since there is no bail-in with $k - 1$ contributing banks that attains welfare losses below W_N, Condition 1 in Lemma 3.5 is satisfied. Thus, $(1, \ldots, 1)$ is an incentive-compatible response to (b, s) by Lemma 3.5. Incentive compatibility implies that for $i \in \mathcal{C}$,

$$b^i = b^i_*(\alpha_{\mathcal{C}}) \leq V^i(L, \ell_*(\alpha_{\mathcal{C}}), \alpha_{\mathcal{C}}) - V^i(p_N, \ell_N, \alpha_N). \tag{53}$$

We will first show that (53) holds with equality. Since $b_*^i(\alpha_{\mathcal{C}}) > 0$ for any $i \in \mathcal{C}$, it follows that

$$0 < V^i(L, \ell_*(\alpha_{\mathcal{C}}), \alpha_{\mathcal{C}}) = \left(c^i + e^i - (1-\alpha_{\mathcal{C}})\ell_*^i(\alpha_{\mathcal{C}}) + (\pi L)^i - w^i - L^i\right)^+$$

and hence $s^i = 0$. Let $IC^i(\alpha, \ell^i)$ be defined as in the proof of Lemma 3.5. As illustrated in Figure 8, $b_*^i(\alpha_{\mathcal{C}}) = IC^i(\alpha, \ell_*^i(\alpha_{\mathcal{C}}))$ if $\ell_*^i(\alpha_{\mathcal{C}}) < e^i$, hence (53) holds with equality. Suppose, therefore, that $\ell_*^i(\alpha) = e^i$. Since $b_*^i(\alpha_{\mathcal{C}}) > 0$ for any $i \in \mathcal{C}$, this implies that the intersection point $\hat{\ell}^i = L^i + w^i - c^i - (\pi p_N)^i + (1-\alpha_N)\ell_N^i$ of $\eta^i(\alpha_{\mathcal{C}}, \ell^i)$ and $IC^i(\alpha_{\mathcal{C}}, \ell^i)$ satisfies $\hat{\ell}^i \geq e^i$. This is equivalent to $V^i(p_N, \ell_N, \alpha_N) = 0$, which implies that $IC^i(\alpha_{\mathcal{C}}, e^i) = \eta^i(\alpha_{\mathcal{C}}, e^i)$. Therefore, (53) holds with equality. Let $\ell^i(b,s)$ denote the amount that bank i liquidates in the bail-in. We have shown

$$b^i - s^i = c^i + e^i - (1-\alpha_{\mathcal{C}})\ell^i(b,s) + (\pi L)^i - w^i - L^i - V^i(p_N, \ell_N, \alpha_N) \tag{54}$$

for any bank $i \in \mathcal{C}$. We will proceed to show that (54) holds also for banks $i \notin \mathcal{C}$. Fix a bank $i \notin \mathcal{C}$ and suppose first that $L^i + w^i - c^i - \alpha_{\mathcal{C}} e^i - (\pi L)^i < 0$. This implies $s^i = 0$ and $V^i(L, \ell, \alpha_{\mathcal{C}}) > 0$ for any ℓ^i. In particular, $V^i(L, \ell_N, \alpha_{\mathcal{C}}) > 0$. Monotonicity of V^i in α and p by Lemma B.2 implies that $V^i(L, \ell_N, \alpha_{\mathcal{C}}) \geq V^i(p_N, \ell_N, \alpha_N)$. If the inequality is strict, then $b_*^i(\alpha) > 0$, violating $i \notin \mathcal{C}$. If the inequality holds with equality, then $\ell_N^i = 0$ and $(\pi L)^i = (\pi p_N)^i$ must hold because we have assumed $\alpha_{\mathcal{C}} > \alpha_N$. It follows that (54) holds with $\ell^i(b,s) = 0$ for such a bank i. Finally, consider $i \notin \mathcal{C}$ with $0 \leq L^i + w^i - c^i - \alpha_{\mathcal{C}} e^i - (\pi L)^i$. Such a bank must liquidate $\ell^i(b,s) = e^i$. Moreover, since $0 \leq L^i + w^i - c^i - \alpha_{\mathcal{C}} e^i - (\pi L)^i < L^i + w^i - c^i - \alpha_N e^i - (\pi p_N)^i$, it follows that $V^i(p_N, \ell_N, \alpha_N) = 0$. We conclude that (54) holds for every bank i. Summing (54) over all banks, we obtain

$$\begin{aligned}
\sum_{i=1}^n (b^i - s^i) &= \sum_{i=1}^n \left(c^i + e^i - w^i - V^i(p_N, \ell_N, \alpha_N)\right) - (1-\alpha_{\mathcal{C}}) \sum_{i=1}^n \ell^i(b,s) \\
&= W_N - (1+\lambda) \sum_{i \in \mathcal{D}_N} \delta^i(p_N, \alpha_N) - (1-\alpha_{\mathcal{C}}) \sum_{i=1}^n \ell^i(b,s),
\end{aligned} \tag{55}$$

where we have used Lemma B.1 in the second equation. Let us denote by

$$\Delta Liq := (1-\alpha_N) \sum_{i=1}^n \ell_N^i - (1-\alpha_{\mathcal{C}}) \sum_{i=1}^n \ell^i(b,s) = \frac{(1-\alpha_{\mathcal{C}})\ln(\alpha_{\mathcal{C}})}{\gamma} - \frac{(1-\alpha_N)\ln(\alpha_N)}{\gamma}$$

the difference in liquidation losses between the default cascade and the bail-in (b,s). Note that $\alpha_{\mathcal{C}} > \alpha_N$ implies $\Delta Liq > 0$. It follows from (6) and (55) that

$$\sum_{i=1}^n (b^i - s^i) = \Delta Liq + (1-\beta) \sum_{i \in \mathcal{D}_N} \left(c^i + \alpha_N e^i + (\pi p_N)^i\right) - \sum_{i \in \mathcal{D}_N} \delta^i(p_N, \alpha_N). \tag{56}$$

Using (55) and the definitions of welfare losses in the default cascade and in a bail-in, we obtain

$$W_N - W_\lambda(b,s,1) = (1+\lambda)\left(\Delta Liq + (1-\beta)\sum_{i\in\mathcal{D}_N}\left(c^i + \alpha_N e^i + (\pi p_N)^i\right)\right) > 0.$$

Consider next the case $\alpha_N \geq \alpha_\mathcal{C}$. Define bail-in (b,s) by setting $b^i = \eta^i(\alpha_N, \ell_N^i)$ and $s^i = s^i(\alpha_N, \ell_N)$ for every bank i. Since the equity value of a bank is non-decreasing in interbank repayments by Lemma B.2, it follows that $V^i(b,s,1) \geq V^i(p_N, \ell_N, \alpha_N)$ for every bank i. In particular, (b,s) is a complete, feasible, and individually incentive-compatible bail-in. Since (α_N, ℓ_N) solves (1), it follows that the induced recovery rate is α_N. Since there is no bail-in with $k-1$ contributing banks that attains welfare losses below W_N, Condition 1 in Lemma 3.5 is satisfied and hence $a = (1,\dots,1)$ is an incentive-compatible response to (b,s) by Lemma 3.5. Observe next that for any bank $i \notin \mathcal{D}_N$, we have $\alpha_N \ell_N^i = (L^i + w^i - c^i - (\pi p_N)^i)^+$. Together with (4), this implies

$$V^i(p_N, \ell_N, \alpha_N) = e^i - \ell_N^i + (c^i + (\pi p_N)^i - w^i - L^i)^+ \geq e^i - \ell_N^i. \tag{57}$$

Using the definitions of η and s, by summing $b^i - s^i$ for all banks i we obtain

$$\begin{aligned}\sum_{i=1}^n (b^i - s^i) &= \sum_{i=1}^n (c^i + \ell_N^i - w^i) - (1-\alpha_N)\sum_{i=1}^n \ell_N^i \\ &\geq \sum_{i=1}^n (c^i + e^i - w^i - V^i(p_N, \ell_N, \alpha_N)) - (1-\alpha_N)\sum_{i=1}^n \ell_N^i,\end{aligned}$$

where we have used that $\ell_N^i = e^i$ and $V^i(p_N, \ell_N, \alpha_N) = 0$ for $i \in \mathcal{D}_N$ and (57) for $i \notin \mathcal{D}_N$ in the second inequality. It now follows analogously as above that

$$W_N - W_\lambda(b,s,1) \geq (1+\lambda)(1-\beta)\sum_{i\in\mathcal{D}_N}\left(c^i + \alpha_N e^i + (\pi p_N)^i\right) \geq 0.$$

In both cases, we have constructed an incentive-compatible bail-in with welfare losses below W_N. It follows from Lemma F.3 that $W_k \leq W_\lambda(b,s,1) \leq W_N$.

For the last statement, let (b,s) be any bail-in from $\Xi_*(\mathcal{C}_m, \alpha^*_{\mathcal{C}_m})$. By Lemma F.1, (b,s) is a complete, feasible, and individually incentive-compatible bail-in. Since $\chi_{\mathcal{C}_m}(\alpha^*_{\mathcal{C}_m}) = 0$ by Lemma F.3, no welfare burning is needed to deter free-riding. Therefore, $(1,\dots,1)$ is an equilibrium response by Lemma 3.5. Since $W_m < W_N$ by definition, $(1,\dots,1)$ is an accepting equilibrium. □

Lemma F.6. *For any set of banks $\mathcal{C}$ and any $i \in \mathcal{C}$, let $W^i_\mathcal{C}(\alpha) := W_P - h_\mathcal{C}(\alpha) + \chi^i_\mathcal{C}(\alpha)$. Let $\tilde{\alpha}_\mathcal{C}$ be defined as in Theorem D.2. The function $W^\dagger_\mathcal{C} := \max_{i\in\mathcal{C}} W^i_\mathcal{C}$ is quasi-convex with its minimum on $[\alpha_\mathcal{C}, 1]$ attained at $\tilde{\alpha}_\mathcal{C}$.*

Proof. Fix a set of banks $\mathcal{C}$. For any $i \in \mathcal{C}$, denote $b^i(\alpha) := b^i_*(\alpha) - \underline{b}^i$ for the sake of brevity. Set $W_0 := W_N - W_P$ so that $\chi^i_\mathcal{C}(\alpha) = \left(W_0 + h_{\mathcal{C}\setminus\{i\}}(\alpha) - g_\alpha(b^i(\alpha))\right)^+$. We begin by computing the partial derivatives of g_α. Abbreviate $\alpha_0(x) := z^{-1}(z(\alpha) + \gamma x)$ so that $g_\alpha(x) = g(\alpha_0(x)) - g(\alpha)$. The partial derivatives of $\alpha_0(x)$ with respect to x and α are

$$\alpha_0'(x) = \frac{\partial \alpha_0(x)}{\partial x} = \frac{\gamma}{z'(\alpha_0(x))}, \qquad \frac{\partial \alpha_0(x)}{\partial \alpha} = \frac{z'(\alpha)}{z'(\alpha_0(x))} = \frac{z'(\alpha)}{\gamma}\alpha_0'(x).$$

Recalling that $g'(\alpha) = (1+\lambda)z'(\alpha)/\gamma - 1/(\gamma\alpha)$ and $z'(\alpha) = 1 + \ln(\alpha)$, the partial derivatives of g_α with respect to x and α are equal to

$$\begin{aligned} \frac{\partial g_\alpha(x)}{\partial x} &= g'(\alpha_0(x))\alpha_0'(x) = (1+\lambda) - \frac{1}{\alpha_0(x)z'(\alpha_0(x))} = (1+\lambda)\Big(1 - \frac{\alpha_{\text{ind}} z'(\alpha_{\text{ind}})}{\alpha_0(x)z'(\alpha_0(x))}\Big), \\ \frac{\partial g_\alpha(x)}{\partial \alpha} &= g'(\alpha_0(x))\frac{\partial \alpha_0(x)}{\partial \alpha} - g'(\alpha) = \frac{1}{\gamma\alpha}\Big(1 - \frac{\alpha z'(\alpha)}{\alpha_0(x)z'(\alpha_0(x))}\Big). \end{aligned} \tag{58}$$

Since z is increasing on $[\alpha_{\text{ind}}, 1]$, it follows that $\alpha_0(x) \geq \alpha \geq \alpha_{\text{ind}}$ for any $\alpha \in [\alpha_{\text{ind}}, 1]$ and any $x \geq 0$. Convexity of z thus implies that both derivatives in (58) are non-negative. We use this fact to show the following two claims.

Claim.

1. On the interval $[\alpha_{\text{ind}}, 1]$, once $\chi^i_{\mathcal{C}}(\alpha)$ reaches 0, it stays 0.
2. On the interval $[\alpha_{\text{ind}}, 1]$, $W^i_{\mathcal{C}}(\alpha)$ is non-increasing where $\chi^i_{\mathcal{C}}(\alpha) > 0$.

To show the first claim, observe that $h_{\mathcal{C}\setminus\{i\}}$ is non-increasing on $[\alpha_{\text{ind}}, 1]$ by Lemma B.4. It is easy to verify from the definition of $b^i_*(\alpha)$ in (14) that $b^i(\alpha)$ is non-decreasing in α. Therefore,

$$\frac{\partial(W_0 + h_{\mathcal{C}\setminus\{i\}}(\alpha) - g_\alpha(b^i(\alpha)))}{\partial \alpha} = h'_{\mathcal{C}\setminus\{i\}}(\alpha) - \frac{\partial g_\alpha(b^i(\alpha))}{\partial \alpha} - \frac{\partial g_\alpha(b^i(\alpha))}{\partial x}\frac{\partial b^i(\alpha)}{\partial \alpha} \tag{59}$$

is non-positive on $[\alpha_{\text{ind}}, 1]$. This shows that $\chi^i_{\mathcal{C}}(\alpha)$ stays 0 once it reaches 0 on the interval $[\alpha_{\text{ind}}, 1]$. For the second claim, observe that for any $\alpha \geq \alpha_{\text{ind}}$, where $\chi^i_{\mathcal{C}}(\alpha)$ is positive, the definition of $W^i_{\mathcal{C}}(\alpha)$ implies that $W^i_{\mathcal{C}}(\alpha) = W_N - \lambda \underline{b}^i - g_\alpha(b^i(\alpha))$, hence

$$\frac{\partial W^i_{\mathcal{C}}(\alpha)}{\partial \alpha} = -\frac{\partial g_\alpha(b^i(\alpha))}{\partial \alpha} - \frac{\partial g_\alpha(b^i(\alpha))}{\partial x}\frac{\partial b^i(\alpha)}{\partial \alpha} \leq 0.$$

Note that the inequality is strict if and only if $b^i(\alpha) > 0$ or $\eta^i(\alpha, 0) = b^i_0$. For α, where $\chi^i_{\mathcal{C}}(\alpha) = 0$, we have $W^i_{\mathcal{C}}(\alpha) = W_N - W_0 + g(\alpha) - \lambda \sum_{i\in\mathcal{C}} \underline{b}^i$, which is increasing on $[\alpha_{\text{ind}}, 1]$. This shows that $W^i_{\mathcal{C}}(\alpha)$ is quasi-convex with and that it is minimized at $\tilde{\alpha}^i_{\mathcal{C}}$.

Let $\tilde{\alpha}_{\mathcal{C}} := \alpha^*_{\mathcal{C}} \vee \max_{i\in\mathcal{C}} \tilde{\alpha}^i_{\mathcal{C}}$ be defined as in Theorem D.2. Since $\tilde{\alpha}_{\mathcal{C}} \geq \tilde{\alpha}^i_{\mathcal{C}}$ for any $i \in \mathcal{C}$, it follows that $\chi^i_{\mathcal{C}}(\tilde{\alpha}_{\mathcal{C}}) = 0$ for any $i \in \mathcal{C}$, hence $W^\dagger_{\mathcal{C}} = W_P - h_{\mathcal{C}}(\alpha)$ on the interval $[\tilde{\alpha}_{\mathcal{C}}, 1]$. Since $\tilde{\alpha}_{\mathcal{C}} \geq \alpha_{\text{ind}}$ by construction, $W^\dagger_{\mathcal{C}}$ is increasing on $[\tilde{\alpha}_{\mathcal{C}}, 1]$. If $\tilde{\alpha}_{\mathcal{C}} = \alpha_{\mathcal{C}}$, then $W^\dagger_{\mathcal{C}}$ is increasing on the entire feasible interval $[\alpha_{\mathcal{C}}, 1]$, hence it is minimized at $\tilde{\alpha}_{\mathcal{C}}$. If $\tilde{\alpha}_{\mathcal{C}} = \alpha_{\text{ind}}$, then $W^\dagger_{\mathcal{C}}$ is globally minimized at $\tilde{\alpha}_{\mathcal{C}}$. Finally, if $\tilde{\alpha}_{\mathcal{C}} = \tilde{\alpha}^{i_0}_{\mathcal{C}}$ for some $i_0 \in \mathcal{C}$, then $W^\dagger_{\mathcal{C}}(\alpha) \geq W^{i_0}_{\mathcal{C}}(\alpha) \geq W^{i_0}_{\mathcal{C}}(\tilde{\alpha}_{\mathcal{C}}) = W^\dagger_{\mathcal{C}}(\tilde{\alpha}_{\mathcal{C}})$. □

Lemma F.7. *$\hat{W}_{\mathcal{C}}(\alpha) = W_P - h_{\mathcal{C}}(\alpha) + \hat{\chi}_{\mathcal{C}}(\alpha)$ is strictly quasi-convex on $[\alpha_{\mathcal{C}}, 1]$.*

Proof. As a first step, we will show again that $\hat{\chi}_{\mathcal{C}}$ stays 0 once it reaches 0. To do that, let $f_i(\alpha) := W_N - \hat{W}_{\mathcal{C}}(\alpha) - \lambda \underline{b}^i$ for any bank $i \in \mathcal{C}$ and define $\mathcal{C}(\alpha) = \{i \in \mathcal{C} \mid f_i(\alpha) > 0\}$. For any α, for which $\hat{\chi}_{\mathcal{C}}(\alpha) > 0$, the definition of $\hat{\chi}_{\mathcal{C}}$ in Lemma A.1 implies that

$$-\frac{z(\alpha)}{\gamma} = \sum_{i \in \mathcal{C}(\alpha)} g_\alpha^{-1}(f_i(\alpha)) \tag{60}$$

For any $\alpha \geq \alpha_{\text{ind}}$, define $\hat{\alpha}_x(\alpha) = g^{-1}(x + g(\alpha))$ for $x \geq 0$ so that $g_\alpha^{-1}(x) = \frac{1}{\gamma}(z(\hat{\alpha}_x(\alpha)) - z(\alpha))$. The partial derivatives of $\hat{\alpha}_x$ are equal to

$$\frac{\partial \hat{\alpha}_x(\alpha)}{\partial x} = \frac{1}{g'(g^{-1}(x + g(\alpha)))} = \frac{1}{g'(\hat{\alpha}_x(\alpha))}, \qquad \frac{\partial \hat{\alpha}_x(\alpha)}{\partial \alpha} = \frac{g'(\alpha)}{g'(\hat{\alpha}_x(\alpha))}. \tag{61}$$

Observe that $\alpha \geq \alpha_{\text{ind}}$ implies that $\hat{\alpha}_x(\alpha) \geq \alpha \geq \alpha_{\text{ind}}$ and hence $\hat{\alpha}_x(\alpha)$ is on the increasing part of g. The partial derivatives of $\hat{\alpha}_x(\alpha)$ are thus positive. The partial derivatives of g_α^{-1} satisfy

$$\frac{\partial g_\alpha^{-1}(x)}{\partial x} = \frac{1}{\gamma} z'(\hat{\alpha}_x(\alpha)) \frac{\partial \hat{\alpha}_x(\alpha)}{\partial x} = \frac{z'(\hat{\alpha}_x(\alpha))}{\gamma g'(\hat{\alpha}_x(\alpha))}, \tag{62}$$

$$\frac{\partial g_\alpha^{-1}(x)}{\partial \alpha} = \frac{1}{\gamma} z'(\hat{\alpha}_x(\alpha)) \frac{\partial \hat{\alpha}_x(\alpha)}{\partial \alpha} - \frac{z'(\alpha)}{\gamma} = g'(\alpha) \frac{\partial g_\alpha^{-1}(x)}{\partial x} - \frac{z'(\alpha)}{\gamma}. \tag{63}$$

At any continuity point of $\mathcal{C}(\alpha)$, the derivative of (60) with respect to α is

$$\begin{aligned} -\frac{z'(\alpha)}{\gamma} &= \sum_{i \in \mathcal{C}(\alpha)} \left(\frac{\partial g_\alpha^{-1}(f_i(\alpha))}{\partial \alpha} - \frac{\partial g_\alpha^{-1}(f_i(\alpha))}{\partial x} \hat{W}'_{\mathcal{C}}(\alpha) \right) \\ &= -|C(\alpha)| \frac{z'(\alpha)}{\gamma} + \sum_{i \in \mathcal{C}(\alpha)} \frac{\partial g_\alpha^{-1}(f_i(\alpha))}{\partial x} \underbrace{(g'(\alpha) - \hat{W}'_{\mathcal{C}}(\alpha))}_{-\hat{\chi}'_{\mathcal{C}}(\alpha)}, \end{aligned} \tag{64}$$

where we have used (63) in the second equation. It follows from (62) and (64) that

$$\hat{\chi}'_{\mathcal{C}}(\alpha) = -\frac{(|\mathcal{C}(\alpha)| - 1) z'(\alpha)}{\sum_{i \in \mathcal{C}(\alpha)} \frac{z'(\hat{\alpha}_{f_i(\alpha)}(\alpha))}{g'(\hat{\alpha}_{f_i(\alpha)}(\alpha))}} < 0. \tag{65}$$

Note that (65) implies that if $\hat{\chi}_{\mathcal{C}}(\alpha) > 0$ for some α, then $\hat{\chi}_{\mathcal{C}}(\alpha') > 0$ for all $\alpha' < \alpha$. The converse is that $\hat{\chi}_{\mathcal{C}}(\alpha) = 0$ for all $\alpha \geq \alpha_0 := \sup\{\alpha \in [\alpha_{\mathcal{C}}, 1] \mid \hat{\chi}_{\mathcal{C}}(\alpha) > 0\}$. Solving (64) for $\hat{W}'_{\mathcal{C}}(\alpha)$ yields

$$\hat{W}'_{\mathcal{C}}(\alpha) = \frac{g'(\alpha)}{\sum_{i \in \mathcal{C}(\alpha)} \frac{z'(\hat{\alpha}_{f_i(\alpha)}(\alpha))}{g'(\hat{\alpha}_{f_i(\alpha)}(\alpha))}} \underbrace{\left(\sum_{i \in \mathcal{C}(\alpha)} \frac{z'(\hat{\alpha}_{f_i(\alpha)}(\alpha))}{g'(\hat{\alpha}_{f_i(\alpha)}(\alpha))} - (|\mathcal{C}(\alpha)| - 1) \frac{z'(\alpha)}{g'(\alpha)} \right)}_{=:\psi(\alpha)}.$$

We will show that ψ is increasing, which implies that $\hat{W}_{\mathcal{C}}$ is strictly quasi-convex on $[\alpha_{\mathcal{C}}, \alpha_0]$. Observe first that $\psi(\alpha)$ is continuous on $[\alpha_{\mathcal{C}}, \alpha_0]$. Indeed, whenever the size of $\mathcal{C}(\alpha)$ changes, $f_i(\alpha) = 0$ for any bank i that enters/leaves $\mathcal{C}(\alpha)$. In particular, $\hat{\alpha}_{f_i(\alpha)}(\alpha) = \alpha$ for such a bank i, hence the two terms appearing in the definition in ψ both increase/shrink by the same amount. Next, we deduce from the quotient rule that

$$\varphi(\alpha) := \frac{\frac{z''(\alpha)g'(\alpha)}{z'(\alpha)} - g''(\alpha)}{\big(g'(\alpha)\big)^2} = -\frac{1 + z'(\alpha)}{\gamma\alpha^2 z'(\alpha)(g'(\alpha))^2}$$

is strictly increasing. It follows from (61) that

$$\frac{\partial \hat{\alpha}_{f_i(\alpha)}(\alpha)}{\partial \alpha} = \frac{f_i'(\alpha) + g'(\alpha)}{g'(\hat{\alpha}_{f_i(\alpha)}(\alpha))} = -\frac{\hat{\chi}_{\mathcal{C}}'(\alpha)}{g'(\hat{\alpha}_{f_i(\alpha)}(\alpha))}.$$

Another application of the quotient rule shows that for any bank $i \in \mathcal{C}(\alpha)$,

$$\begin{aligned}\frac{\partial}{\partial \alpha}\frac{z'(\hat{\alpha}_{f_i(\alpha)}(\alpha))}{g'(\hat{\alpha}_{f_i(\alpha)}(\alpha))} &= -\hat{\chi}_{\mathcal{C}}'(\alpha)\frac{z''(\hat{\alpha}_{f_i(\alpha)}(\alpha)) - \frac{z'(\hat{\alpha}_{f_i(\alpha)}(\alpha))}{g'(\hat{\alpha}_{f_i(\alpha)}(\alpha))}g''(\hat{\alpha}_{f_i(\alpha)}(\alpha))}{\big(g'(\hat{\alpha}_{f_i(\alpha)}(\alpha))\big)^2} \\ &= -\hat{\chi}_{\mathcal{C}}'(\alpha)\frac{z'(\hat{\alpha}_{f_i(\alpha)}(\alpha))}{g'(\hat{\alpha}_{f_i(\alpha)}(\alpha))}\varphi(\hat{\alpha}_{f_i(\alpha)}(\alpha)) \end{aligned} \tag{66}$$

Summing (66) for every $i \in \mathcal{C}(\alpha)$, using the fact that $\varphi(\hat{\alpha}_{f_i(\alpha)}(\alpha)) > \varphi(\alpha)$ for any bank $i \in \mathcal{C}(\alpha)$ by monotonicity of φ, and using (65) shows that $\psi'(\alpha) > 0$. In particular, $\hat{W}_{\mathcal{C}}$ is strictly quasi-convex on $[\alpha_{\mathcal{C}}, \alpha_0]$. If $\alpha_0 > \alpha_{\text{ind}}$, then $\hat{W}_{\mathcal{C}}$ is strictly quasi-convex on $[\alpha_{\mathcal{C}}, 1]$ because $\hat{W}_{\mathcal{C}} = W_P - h_{\mathcal{C}}(\alpha)$ is strictly increasing on $[\alpha_{\text{ind}}, 1]$. If $\alpha_0 < \alpha_{\text{ind}}$, then $\hat{W}_{\mathcal{C}}$ is strictly decreasing on $[\alpha_{\mathcal{C}}, \alpha_0]$ by (65). Since $\hat{W}_{\mathcal{C}} = W_P - h_{\mathcal{C}}(\alpha)$ is strictly convex on $[\alpha_0, 1]$, the statement follows. □

Proof of Theorem D.2. Observe first that $\hat{W}_{\mathcal{C}}$ is strictly quasi-convex by Lemma F.7, hence $\hat{W}_{\mathcal{C}}$ is uniquely minimized at $\hat{\alpha}_{\mathcal{C}}$. Let $W_{\mathcal{C}}^{\dagger}$ be defined as in Lemma F.6 so that $W_{\mathcal{C}} = \hat{W}_{\mathcal{C}} \vee W_{\mathcal{C}}^{\dagger}$. For statement (i), suppose first that $\hat{\chi}_{\mathcal{C}}(\tilde{\alpha}_{\mathcal{C}}) = 0$. Then

$$W_{\mathcal{C}}(\tilde{\alpha}_{\mathcal{C}}) = W_{\mathcal{C}}^{\dagger}(\tilde{\alpha}_{\mathcal{C}}) \leq W_{\mathcal{C}}^{\dagger}(\alpha) \leq W_{\mathcal{C}}(\alpha)$$

for any $\alpha \in [\alpha_{\mathcal{C}}, 1]$, hence $W_{\mathcal{C}}$ is minimized at $\tilde{\alpha}_{\mathcal{C}}$. Suppose, therefore, that $\hat{\chi}_{\mathcal{C}}(\tilde{\alpha}_{\mathcal{C}}) > 0$. It will be convenient to denote $\varphi^i(\alpha, \ell) = c^i + \alpha\ell^i + (\pi L)^i - w^i - L^i$ so that $\eta^i(\alpha, \ell^i) = (\varphi^i(\alpha, \ell))^+$. As in the proof of Lemma A.1, let $IC^i(\alpha, \ell) = \big(\pi(L - p_N)\big)^i + (1 - \alpha_N)\ell_N^i - (1 - \alpha)\ell^i$ so that

$$b_*^i(\alpha) = \max_{\ell^i \in [0, e^i]} \min\Big(IC^i(\alpha, \ell), \eta^i(\alpha, \ell)\Big). \tag{67}$$

Let $\ell_*^i(\alpha)$ denote the maximizing liquidation decision in (67). Since $\eta^i(\alpha, \ell_*^i(\alpha)) \geq b_*^i(\alpha) \geq b^i > 0$ for any contributing bank i, it follows that $\eta^i(\alpha, \ell_*^i(\alpha)) = \varphi^i(\alpha, \ell_*^i(\alpha))$, hence the maximum in (67) is attained either at the intersection point of $IC^i(\alpha, \ell)$ and $\varphi^i(\alpha, \ell)$ or at the boundary of $[0, e^i]$.

The intersection point $\hat{\ell}^i$ of $IC^i(\alpha, \ell)$ and $\varphi^i(\alpha, \ell)$ is given by

$$\hat{\ell}^i = L^i + w^i - c^i - \pi p_N^i + (1 - \alpha_N)\ell_N^i.$$

and does not depend on α, hence neither does $\ell_*^i = \max(\min(\hat{\ell}^i, e^i), 0)$. It follows from the definition of $\hat{\chi}_{\mathcal{C}}(\alpha)$, that $\hat{\chi}_{\mathcal{C}}(\alpha) > \max_{i\in\mathcal{C}} \chi_{\mathcal{C}}^i(\alpha)$ if and only if the liquidation required for each bank $i \in \mathcal{C}$ to contribute $b^i(\alpha)$ depresses the asset recovery rate below α, that is, if and only if $\alpha(\ell_*) < \alpha$, where we set $\ell_*^i = 0$ for any non-contributing bank i. Since we have assumed that $\hat{\chi}_{\mathcal{C}}(\tilde{\alpha}_{\mathcal{C}}) > 0$, this implies that $\alpha(\ell_*) < \tilde{\alpha}_{\mathcal{C}}$. Let us distinguish two cases. If $\hat{\alpha}_{\mathcal{C}} < \alpha(\ell_*)$, then $\hat{W}_{\mathcal{C}}$ is strictly increasing to the right of $\alpha(\ell_*)$ by Lemma F.7 and $W_{\mathcal{C}}^\dagger$ is non-increasing to the left of $\alpha(\ell_*)$ as in the proof of Lemma F.6, hence $W_{\mathcal{C}}$ is minimized at $\alpha(\ell_*)$. If $\alpha(\ell_*) \le \hat{\alpha}_{\mathcal{C}} < \tilde{\alpha}_{\mathcal{C}}$, then $W_{\mathcal{C}}(\hat{\alpha}_{\mathcal{C}}) = \hat{W}_{\mathcal{C}}(\hat{\alpha}_{\mathcal{C}}) \le \hat{W}_{\mathcal{C}}(\alpha) \le W_{\mathcal{C}}(\alpha)$ concludes the proof of statement (i). Statement (ii) is proven in Lemma F.3.

For the last statement, suppose first that bail-ins with contributing banks in $\mathcal{C}$ may arise in a subgame Pareto-efficient equilibrium. It follows from Lemma F.7 that $\alpha(\mathcal{C})$ is the unique minimizer of $W_{\mathcal{C}}$ if $\alpha(\mathcal{C}) = \hat{\alpha}_{\mathcal{C}}$. If $\alpha(\mathcal{C}) = \tilde{\alpha}_{\mathcal{C}}$ or $\alpha(\mathcal{C}) = \alpha(\ell_*)$, then $W_{\mathcal{C}}^\dagger(\alpha)$ defined in Lemma F.6 may be constant to the left of $\alpha(\mathcal{C})$. Suppose that there exists $\alpha < \alpha(\mathcal{C})$, for which $W_{\mathcal{C}}(\alpha) = W_{\mathcal{C}}(\alpha(\mathcal{C}))$. Bail-ins in $\Xi(\mathcal{C}, \alpha)$ differ from bail-ins in $\Xi(\mathcal{C}, \alpha(\mathcal{C}))$ in that banks in $\mathcal{C}$ contribute a larger amount and the regulator throws those contributions away. These bail-ins are subgame-Pareto dominated by bail-ins in $\Xi(\mathcal{C}, \alpha(\mathcal{C}))$. Thus, the asset recovery rate in a subgame Pareto efficient equilibrium with contributing banks in $\mathcal{C}$ is unique and it is equal to $\alpha(\mathcal{C})$.

For any two sets of banks $\mathcal{C}_1 \neq \mathcal{C}_2$, let $\Omega(\mathcal{C}_1, \mathcal{C}_2)$ denote the set of parameters, for which bail-ins from $\Xi(\mathcal{C}_1, \alpha(\mathcal{C}_1))$ as well as $\Xi(\mathcal{C}_2, \alpha(\mathcal{C}_2))$ are subgame Pareto efficient such that $\mathcal{C}_1$ and $\mathcal{C}_2$ do not differ only by banks i with $b_0^i = 0$. Since the regulator gets to choose which bail-in to propose, this imposes $W_{\mathcal{C}_1}(\alpha(\mathcal{C}_1)) = W_{\mathcal{C}_2}(\alpha(\mathcal{C}_2))$ on $\Omega(\mathcal{C}_1, \mathcal{C}_2)$. We will show that $\Omega^c(\mathcal{C}_1, \mathcal{C}_2)$ contains a subset that is open and dense in the space of all parameters. It follows from the definition of m that $\Omega(\mathcal{C}_1, \mathcal{C}_2) = \emptyset$ if either $|\mathcal{C}_1| < m$ or $|\mathcal{C}_2| < m$, hence the statement is trivially satisfied. Consider next the case, in which $|\mathcal{C}_1| = |\mathcal{C}_2| = m$. By statement (ii), $\alpha(\mathcal{C}_i) = \alpha_{\mathcal{C}_i}^*$ and $\chi_{\mathcal{C}_i}(\alpha(\mathcal{C}_i)) = 0$ for $i = 1, 2$. Thus, $W_{\mathcal{C}_1}(\alpha(\mathcal{C}_1)) = W_{\mathcal{C}_2}(\alpha(\mathcal{C}_2))$ if and only if

$$\lambda \sum_{i\in\mathcal{C}_1} \underline{b}^i - g(\alpha_{\mathcal{C}_1}^*) - \lambda \sum_{i\in\mathcal{C}_2} \underline{b}^i + g(\alpha_{\mathcal{C}_2}^*) = 0. \tag{68}$$

For any set of banks $\mathcal{D}$, let us denote by $\Omega(\mathcal{D})$ the closure of the set of parameters, for which $\mathcal{D}_N = \mathcal{D}$. Note that for a fixed set of defaulting banks, $\underline{b}^i$ is continuous in the model parameters. Lemma E.2 implies that $\alpha_{\mathcal{C}_i}$ is continuous in the parameters as well. Since also α_{ind} and g are continuous, the left-hand side of (68) is continuous on $\Omega(\mathcal{D})$, hence $\Omega(\mathcal{C}_1, \mathcal{C}_2) \cap \Omega(\mathcal{D})$ is closed. Next, we show that any set of parameters in $\Omega(\mathcal{C}_1, \mathcal{C}_2) \cap \Omega(\mathcal{D})$ can be approximated by a set of parameters in $\Omega^c(\mathcal{C}_1, \mathcal{C}_2) \cap \Omega(\mathcal{D})$. Consider first the case, in which $\alpha_{\mathcal{C}_1}^* = \alpha_{\mathcal{C}_2}^* = \alpha_{\text{ind}}$. For $k = 1, 2$, define

$$C_k^+ := \Big\{ i \in \mathcal{C}_k \;\Big|\; \ell_*^i > 0 \Big\}, \qquad C_k^- := \Big\{ i \in \mathcal{C}_k \;\Big|\; s^i(\alpha_{\mathcal{C}_k}^*, e) > 0 \Big\}.$$

Suppose first that $\alpha^*_{\mathcal{C}_1} = \alpha^*_{\mathcal{C}_2} = \alpha_{\text{ind}}$. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ differ not only by banks with $b^i_0 = 0$, without loss of generality, there exists $i \in \mathcal{C}_1 \setminus \mathcal{C}_2$ with $b^i_0 > 0$. If there exists such a bank $i \in \mathcal{C}_1^+$, then $\underline{b}^i = b^i_0 > 0$, hence an increase in c^i increases $\underline{b}^i$ without affecting α_{ind} or $\underline{b}^j$ for any other bank j. Moreover, for a sufficiently slight increase of c^i, the set $\mathcal{D}_N$ stays constant. Suppose, therefore, that there exists no such bank $i \in \mathcal{C}_1^+$. If there exists a bank in $j \in \mathcal{D}_N$ such that

$$\frac{\partial}{\partial L^j} \sum_{i \in \mathcal{C}_1} \underline{b}^i \neq \frac{\partial}{\partial L^j} \sum_{i \in \mathcal{C}_2} \underline{b}^i \tag{69}$$

then an increase in L^j leaves $\mathcal{D}_N$ constant and breaks equality in (68). If there exists no such bank j, let us change π slightly such that $\mathcal{D}_N$ is unaffected and (69) holds for some $j \in \mathcal{D}_N$ after the change to π. If the inequality in (68) is already broken by that change, we are done, otherwise we simultaneously increase L^j as we vary π, which breaks equality in (68). If $\alpha^*_{\mathcal{C}_k} \neq \alpha_{\text{ind}}$ for at least one $k \in \{1, 2\}$, it is easier to break (68) because there are more variables to play with. We conclude that each $\Omega^c(\mathcal{C}_1, \mathcal{C}_2) \cap \Omega(\mathcal{D})$ is open and dense in $\Omega(\mathcal{D})$. Taking the union over the finitely many possibilities for $\mathcal{D}$ shows that $\Omega^c(\mathcal{C}_1, \mathcal{C}_2)$ contains an open and dense set of parameters.

Consider next the case, in which $|\mathcal{C}_k| > m$ for $k = 1, 2$. Since the no-free-riding constraint is binding for both sets of banks, it follows that $W_{\mathcal{C}_1}(\alpha(\mathcal{C}_1)) = W_{\mathcal{C}_2}(\alpha(\mathcal{C}_2))$ if and only if

$$\min_{i \in \mathcal{C}_1}\big(\lambda \underline{b}^i + g_{\alpha(\mathcal{C}_1)}(b^i(\alpha(\mathcal{C}_1)))\big) - \min_{i \in \mathcal{C}_2}\big(\lambda \underline{b}^i + g_{\alpha(\mathcal{C}_2)}(b^i(\alpha(\mathcal{C}_2)))\big) = 0. \tag{70}$$

It follows from the definition of $\chi^i_{\mathcal{C}_k}$ and $\hat{\chi}_{\mathcal{C}_k}$ in Lemma A.1 that $\tilde{\alpha}_{\mathcal{C}_k}$ and $\hat{\alpha}_{\mathcal{C}_k}$ is continuous for $k = 1, 2$. Since also $\alpha(\ell_*(\mathcal{C}_k))$ is continuous, it follows again from continuity of the left-hand side of (70) that $\Omega(\mathcal{C}_1, \mathcal{C}_2) \cap \Omega(\mathcal{D})$ is closed for each set $\mathcal{D}$. For each $k = 1, 2$, let $i_k \in \mathcal{C}_k$ denote a bank that attains the respective minimum in (70). Observe that it is not possible that each i_k is unique and that $i_1 = i_2$ because the regulator would be better off replacing i_1 with a bank $i \in \mathcal{C}_2 \setminus \mathcal{C}_1$. Thus, we can choose $i_1 \neq i_2$ and proceed as above: approximate some parameter that breaks the inequality in (70). If that does not exist, combine that change with a change in the network structure.

Finally, consider the case, in which $|\mathcal{C}_1| = m$ and $|\mathcal{C}_2| > m$. Then the no-free-riding constraint is binding in $\mathcal{C}_2$ but not in $\mathcal{C}_1$. Note that $W_{\mathcal{C}_1}(\alpha(\mathcal{C}_1)) = W_{\mathcal{C}_2}(\alpha(\mathcal{C}_2))$ if and only if

$$W_P - g(\alpha_P) + g(\alpha(\mathcal{C}_1)) - \lambda \sum_{i \in \mathcal{C}_1} \underline{b}^i = W_N - \min_{i \in \mathcal{C}_2}\big(\lambda \underline{b}^i + g_{\alpha(\mathcal{C}_2)}(b^i(\alpha(\mathcal{C}_2)))\big).$$

We can proceed exactly in the same way as above that $\Omega^c(\mathcal{C}_1, \mathcal{C}_2)$ is open and dense. □

Proof of Corollary D.3. This follows from the construction of $\alpha(\ell_*)$ and $\hat{\alpha}_{\mathcal{C}}$. The former is the intersection point of $\hat{W}_{\mathcal{C}}$ and $W^\dagger_{\mathcal{C}}$ and the latter is the minimum of $\hat{W}_{\mathcal{C}}$ if $\hat{W}_{\mathcal{C}}(\hat{\alpha}_{\mathcal{C}}) \geq W^\dagger_{\mathcal{C}}(\hat{\alpha}_{\mathcal{C}})$. □

G Comparative Statics Results

G.1 Complete Bailout

Many of our comparative statics results will utilize the dependence of α_L on the underlying parameters. For the sake of reference, we isolate the partial derivatives of α_L in the following lemma. Recall that we denote by $\mathcal{F}$ the set of fundamentally defaulting banks.

Lemma G.1. *Let $\xi(\gamma, s_0, e) := 1/(1+\ln(\alpha_L)+\gamma\sum_{i\in\mathcal{F}} e^i)$. Then $\xi(\gamma, s_0, e) \geq 1$. Moreover, α_L is continuous and it is differentiable in γ, s_0, and e where $\mathcal{F}$ is constant with partial derivatives*

$$\frac{\partial\alpha_L}{\partial\gamma} = \frac{\alpha_L\ln(\alpha_L)}{\gamma}\xi, \qquad \frac{\partial\alpha_L}{\partial s_0^i} = -\gamma\xi 1_{\{i\notin\mathcal{F}\}}, \qquad \frac{\partial\alpha_L}{\partial e^i} = -\gamma\alpha_L\xi 1_{\{i\in\mathcal{F}\}}. \tag{71}$$

Proof. Continuity and differentiability where $\mathcal{F}$ is constant follows from Lemma E.1 for $p = L$. Since $\ell(L, \alpha_L) = \min(\frac{s_0}{\alpha_L}, e)$, we can write $\alpha_L = \exp(-\gamma\sum_{i\notin\mathcal{F}}\frac{s_0^i}{\alpha_L} - \gamma\sum_{i\in\mathcal{F}} e^i)$. Implicit differentiation with respect to γ at a differentiability point yields

$$\frac{\partial\alpha_L}{\partial\gamma} = \alpha_L\left(-\sum_{i\notin\mathcal{F}}\frac{s_0^i}{\alpha_L} - \sum_{i\in\mathcal{F}} e^i + \frac{\gamma}{\alpha_L^2}\sum_{i\notin\mathcal{F}} s_0^i\frac{\partial\alpha_L}{\partial\gamma}\right) = \frac{\alpha_L\ln(\alpha_L)}{\gamma} - \left(\ln(\alpha_L) + \gamma\sum_{i\in\mathcal{F}} e^i\right)\frac{\partial\alpha_L}{\partial\gamma}, \tag{72}$$

where in the second equation we have used that

$$-\sum_{i\notin\mathcal{F}}\frac{s_0^i}{\alpha_L} - \sum_{i\in\mathcal{F}} e^i = \frac{\ln(\alpha_L)}{\gamma} \qquad \text{and} \qquad \frac{\gamma}{\alpha_L}\sum_{i\notin\mathcal{F}} s_0^i = -\ln(\alpha_L) - \gamma\sum_{i\in\mathcal{F}} e^i. \tag{73}$$

Solving (72) for $\frac{\partial\alpha_L}{\partial\gamma}$ yields the desired result. For the partial derivatives with respect to s_0^i and e^i, observe that α_L does not depend on s_0^i for $i \in \mathcal{F}$ and it does not depend on e^i for $i \notin \mathcal{F}$. For $i \notin \mathcal{F}$, using implicit differentiation and the identities in (73), we obtain

$$\frac{\partial\alpha_L}{\partial s_0^i} = \alpha_L\left(-\frac{\gamma}{\alpha_L} + \frac{\gamma}{\alpha_L^2}\sum_{i\notin\mathcal{F}} s_0^i\frac{\partial\alpha_L}{\partial s_0^i}\right) = -\gamma - \left(\ln(\alpha_L) + \gamma\sum_{i\in\mathcal{F}} e^i\right)\frac{\partial\alpha_L}{\partial s_0^i} = -\gamma\xi(\gamma, s_0, e).$$

It follows in the same way that for $i \in \mathcal{F}$, the partial derivative with respect to e^i is

$$\frac{\partial\alpha_L}{\partial e^i} = \alpha_L\left(-\gamma + \frac{\gamma}{\alpha_L^2}\sum_{i\notin\mathcal{F}} s_0^i\frac{\partial\alpha_L}{\partial e^i}\right) = -\gamma\alpha_L - \left(\ln(\alpha_L) + \gamma\sum_{i\in\mathcal{F}} e^i\right)\frac{\partial\alpha_L}{\partial e^i} = -\gamma\alpha_L\xi(\gamma, s_0, e).$$

It remains to show that $\xi(\gamma, s_0, e) \geq 1$. It follows from Lemma C.2 that $\alpha_L \geq \frac{1}{\mathrm{e}}$ and hence $1 + \ln(\alpha_L) + \gamma\sum_{i\in\mathcal{F}} e^i > 0$, where we have used that $\mathcal{F}$ is non-empty by assumption. The second identity in (73) implies that $1 + \ln(\alpha_L) + \gamma\sum_{i\in\mathcal{F}} e^i \leq 1$ and, therefore, $\xi(\gamma, s_0, e) \geq 1$. □

Recall that the results in Section 4 are derived under the assumptions that $\lambda > 0$, $\gamma > 0$, $e^i > 0$ for every bank i, and that there exists at least one fundamentally defaulting bank.

Lemma G.2. *There exist (possibly infinite) thresholds $\lambda_*, \gamma_* > 0$ and $e^i_*, s^i_* \geq 0$ for each bank i such that the following conditions hold:*

(i) $\alpha_P = \alpha_L$ for $\lambda > \lambda_*$, $\gamma \leq \gamma_*$, $e^i \leq e^i_*$, and $s^i_0 \leq s^i_*$.

(ii) $\alpha_P = \alpha_{\text{ind}}$ for $\lambda \leq \lambda_*$, $\gamma > \gamma_*$, $e^i > e^i_*$, and $s^i_0 > s^i_*$.

We re-iterate from Footnote 30 that the thresholds depend on the other parameters of the model, that is, γ_* depends on λ, e, and s_0, the threshold e^i_* depends on γ, λ, e^{-i}, and s_0, etc.

Proof. The statement will follow from the monotonicity properties of α_L and α_{ind}, and the fact that $\alpha_P = \max(\alpha_{\text{ind}}, \alpha_L)$ by Lemma 3.2. Observe first that α_L is constant in λ and that $\alpha_L < 1$ since there is at least one fundamentally defaulting bank i, liquidating a positive amount $e^i > 0$ at elasticity $\gamma > 0$. Since α_{ind} is decreasing in λ with $\lim_{\lambda \to 0} \alpha_{\text{ind}} = 1 > \alpha_L$, it follows that there exists $\lambda_* \in (0, \infty]$ such that $\alpha_P = \alpha_{\text{ind}}$ for $\lambda \leq \lambda_*$ and $\alpha_P = \alpha_L$ if $\lambda > \lambda_*$.

For the threshold results with respect to γ, e^i, and s^i_0, observe first that α_{ind} does not depend on these parameters. Since α_L is continuous and non-increasing in γ, e^i, and s^i_0 by Lemma G.1, it follows that there exist $\gamma_*, e^i_*, s^i_* \in [0, \infty]$ such that $\alpha_P = \alpha_L$ below these thresholds and $\alpha_P = \alpha_{\text{ind}}$ above these thresholds. Finally, $\gamma_* > 0$ since $\lim_{\gamma \to 0} \alpha_L = 1 > \alpha_{\text{ind}}$. $\square$

Lemma G.3. *Let λ_*, γ_*, e^i_*, and s^i_* be as Lemma G.2. There exist finite constants $s^i_1 \leq s^i_*$ and $e^i_1 \leq e^i_*$ such that the following statements hold:*

(i) For $\lambda \leq \lambda_*$, $\gamma \leq \gamma_*$, $e^i \leq e^i_1$, and $s^i_0 \leq s^i_1$, the asset recovery rate α_P is decreasing.

(ii) For $\lambda > \lambda_*$, $\gamma > \gamma_*$, $e^i > e^i_1$, and $s^i_0 > s^i_1$, the asset recovery rate α_P is constant.

Finally, α_P is differentiable almost everywhere and $\alpha_P > \frac{1}{\mathrm{e}}$.

Proof. Note that a bank i is in $\mathcal{F}$ if and only if e^i is sufficiently small or s^i_0 is sufficiently large. Let $\hat{e}^i$ and $\hat{s}^i_0$ be the corresponding thresholds and set $e^i_1 := \min(\hat{e}^i, e^i_*)$ and $s^i_1 := \min(\hat{s}^i_0, s^i_*)$. Statements (i) and (ii) now follow straight from Lemmas G.1 and G.2. Lemma B.4 implies that $\alpha_P \geq \alpha_{\text{ind}} > \frac{1}{\mathrm{e}}$ and differentiability almost everywhere follows from Lemma G.1. $\square$

Lemma G.4. *The total subsidies in the optimal complete bailout are increasing in γ and in s^i_0 for each i. Moreover, for λ_* as in Lemma G.2 and e^i_1 as in Lemma G.3, the following statements hold:*

(i) For $\lambda \leq \lambda_*$ and $e^i \leq e^i_1$, the total subsidies S are decreasing in γ and e^i.

(ii) For $\lambda > \lambda_*$, and $e^i > e^i_1$, the total subsidies S are constant.

Proof. By Lemma 3.2, the total subsidies S in the optimal public bailout are equal to

$$S = \sum_{i=1}^{n} s^i_0 + \frac{\alpha_P \ln(\alpha_P)}{\gamma}. \tag{74}$$

Because the total subsidies are continuous and differentiable almost everywhere, the total subsidies are weakly differentiable. Taking the (weak) partial derivatives with respect to λ and e^i yields

$$\frac{\partial S}{\partial \lambda} = \frac{1+\ln(\alpha_P)}{\gamma}\frac{\partial \alpha_P}{\partial \lambda}, \qquad \frac{\partial S}{\partial e^i} = \frac{1+\ln(\alpha_P)}{\gamma}\frac{\partial \alpha_P}{\partial e^i}. \tag{75}$$

Since $\alpha_P > \frac{1}{\mathrm{e}}$ by Lemma G.3, Lemma G.3 in conjunction with (75) shows the statements for λ and e^i. Taking the (weak) partial derivative of (74) with respect to γ, we obtain

$$\frac{\partial S}{\partial \gamma} = -\frac{\alpha_P \ln(\alpha_P)}{\gamma^2} + \frac{1+\ln(\alpha_P)}{\gamma}\frac{\partial \alpha_P}{\partial \gamma}. \tag{76}$$

Lemma G.3 implies that S is strictly increasing in γ if $\gamma > \gamma_*$. If $\gamma \le \gamma_*$, then $\alpha_P = \alpha_L$ by Lemma G.2. Thus, Lemma G.1, the second identity in (73), and (76) together imply that

$$\frac{\partial S}{\partial \gamma} = \frac{\alpha_L \ln(\alpha_L)\xi}{\gamma^2}\left(\ln(\alpha_L) - \frac{\gamma}{\alpha_L}\sum_{i\notin\mathcal{F}} s_0^i\right) > 0,$$

where we have used the fact that $\alpha_L < 1$. Finally, we compute the (weak) partial derivatives of S with respect to s_0^i. Taking the partial derivative in (74), we obtain

$$\frac{\partial S}{\partial s_0^i} = 1 + \frac{1+\ln(\alpha_P)}{\gamma}\frac{\partial \alpha_P}{\partial s_0^i}. \tag{77}$$

It follows from Lemma G.3 that S is strictly increasing for $s_0^i > s_1^i$. For $s_0^i \le s_1^i$, note that $\alpha_P = \alpha_L$ and hence $i \notin \mathcal{F}$ by Lemma G.1. Therefore, Lemma G.1 shows that

$$\frac{\partial S}{\partial s_0^i} = 1 - (1+\ln(\alpha_L))\xi(\gamma, s_0, e) = \frac{\gamma\sum_{i\in\mathcal{F}} e^i}{1+\ln(\alpha_L)+\gamma\sum_{i\in\mathcal{F}} e^i} > 0,$$

were we have used that the set of fundamentally defaulting banks is non-empty. □

Lemma G.5. *Let e_1^i be defined as in Lemma G.3. Welfare losses W_P in the optimal bailout are increasing in λ, γ, and s_0^i for any bank i, and W_P is decreasing for $e^i \le e_1^i$ and constant for $e^i > e_1^i$.*

Proof. Lemma 3.1 implies that

$$W_P = \lambda\sum_{i=1}^n s_0^i + g(\alpha_P) \tag{78}$$

for $\alpha_P = \max(\alpha_{\text{ind}}, \alpha_L)$. Welfare losses are thus differentiable almost everywhere. The weak partial derivative of (78) with respect to λ is

$$\frac{\partial W_P}{\partial \lambda} = \sum_{i=1}^n s_0^i + \frac{\alpha_P\ln(\alpha_P)}{\gamma} + g'(\alpha_P)\frac{\partial \alpha_P}{\partial \lambda} = \sum_{i=1}^n s_0^i + \frac{\alpha_P\ln(\alpha_P)}{\gamma}, \tag{79}$$

where we have used that $g'(\alpha_{\text{ind}}) = 0$ and that α_L is constant in λ. Together with Lemma 3.2, this implies that

$$\frac{\partial W_P}{\partial \lambda} = \sum_{i=1}^{n} s_0^i + \frac{\alpha_P \ln(\alpha_P)}{\gamma} \geq \sum_{i=1}^{n} s_L^i > 0,$$

where we have used in the last inequality that the set of fundamentally defaulting banks is non-empty. Similarly as above, taking the partial derivative in (78) with respect to γ, we obtain

$$\frac{\partial W_P}{\partial \gamma} = -\frac{g(\alpha_P)}{\gamma} + g'(\alpha_P)\frac{\partial \alpha_P}{\partial \gamma}. \tag{80}$$

If $\gamma \geq \gamma_*$, then $\alpha_P = \alpha_{\text{ind}}$ by Lemma G.2, for which $g(\alpha_{\text{ind}}) < 0$ and $g'(\alpha_{\text{ind}}) = 0$, hence W_P is increasing. If $\gamma < \gamma_*$, then $\alpha_P = \alpha_L$, hence Lemma G.1 implies that

$$\frac{\partial W_P}{\partial \gamma} = \left(\frac{\ln^2(\alpha_L)}{\gamma^2} - g(\alpha_L)\sum_{i \in \mathcal{F}} e^i\right)\xi > 0. \tag{81}$$

For the sensitivity with respect to e^i, note that α_P lies in the interval $[\alpha_{\text{ind}}, 1]$, on which g is increasing. The statement thus follows from Lemma G.3 and

$$\frac{\partial W_P}{\partial e^i} = g'(\alpha_P)\frac{\partial \alpha_P}{\partial e^i}. \tag{82}$$

Taking the partial derivative of (78) with respect to s_0^i yields

$$\frac{\partial W_P}{\partial s_0^i} = \lambda + g'(\alpha_P)\frac{\partial \alpha_P}{\partial s_0^i} \tag{83}$$

If $\alpha_P = \alpha_{\text{ind}}$, then $g'(\alpha_P) = 0$, hence W_P is increasing in s_0^i for any bank i. If $\alpha_P = \alpha_L$, then Lemma G.1 implies that $\frac{\partial \alpha_P}{\partial s_0^i} = 0$ for $i \in \mathcal{F}$, hence W_P is increasing in s_0^i for such a bank i. For a bank $i \notin \mathcal{F}$, Lemma G.1 in conjunction with (83) implies that

$$\frac{\partial W_P}{\partial s_0^i} = \lambda - (1+\lambda)(1+\ln(\alpha_L))\xi + \frac{\xi}{\alpha_L} \geq (1+\lambda)\xi\gamma\sum_{i \in \mathcal{F}} e^i > 0. \qquad \square$$

Proof of Lemma 4.1. The statement follows from Lemmas G.3, G.4, and G.5. $\square$

G.2 Default Cascade

It will be convenient to denote by $\mathcal{D}_N := \mathcal{D}(p_N, \ell_N, \alpha_N)$ the set of defaulting banks in absence of intervention, by $\mathcal{C}_N := \{i \in \mathcal{D}_N \mid \delta^i(p_N, \alpha_N) = 0\}$, the set of defaulting banks which are able to repay a positive amount to their junior creditors, by $\mathcal{S}_N = \mathcal{D}_N^c$ the set of solvent banks, and by $\mathcal{I}_N := \{i \mid 0 < \ell_N^i < e^i\}$ the set of solvent but illiquid banks. Before we get into the sensitivity analysis, we address continuity and differentiability of the default cascade. It will be useful to introduce the following subvector/submatrix notation. For any two sets of banks $\mathcal{I}$ and $\mathcal{C}$ and any vector $x \in \mathbb{R}^n$, let $x^{\mathcal{C}}$ denote the subvector with entries in $\mathcal{C}$ and let $\pi^{\mathcal{I},\mathcal{C}}$ denote the submatrix of π with row and column indices in $\mathcal{I}$ and $\mathcal{C}$, respectively.

Lemma G.6. *p_N and α_N are differentiable in L, π, c, w, e, β, and γ where $\mathcal{D}_N$, $\mathcal{C}_N$, and $\mathcal{I}_N$ are constant. Moreover, p_N and α_N are continuous where $\mathcal{D}_N$ is constant. Finally, p_N is differentiable in α_N where $\mathcal{D}_N$, $\mathcal{C}_N$, and $\mathcal{I}_N$ are constant.*

Proof. Observe that we can write p_N in subvector form as $p_N^{\mathcal{S}_N} = L^{\mathcal{S}_N}$, $p_N^{\mathcal{D}_N \setminus \mathcal{C}_N} = 0$, and

$$p_N^{\mathcal{C}_N} = (I - \beta\pi^{\mathcal{C}_N,\mathcal{C}_N})^{-1}(\beta c^{\mathcal{C}_N} + \beta\alpha_{p_N} e^{\mathcal{C}_N} + \beta\pi^{\mathcal{C}_N,\mathcal{S}_N} L^{\mathcal{S}_N} - w^{\mathcal{C}_N}). \tag{84}$$

Since α_p is differentiable in p by Lemma E.1, this shows differentiability of p_N where $\mathcal{D}_N$, $\mathcal{C}_N$, $\mathcal{S}_N$, and $\mathcal{I}_N$ are constant. Then, differentiability of $\alpha_N = \alpha_{p_N}$ follows from Lemma E.1. It follows from (2) and (3) that p_N and α_N are continuous unless the set $\mathcal{D}_N$ of defaulting banks changes. □

Lemma G.7. *For each i, there exists $0 \leq e_1^i \leq e_*^i$ and $0 \leq c_1^i \leq c_*^i \leq c_2^i$ such that i defaults for $e^i < e_*^i$ or $c^i < c_*^i$ and it is solvent for $e^i \geq e_*^i$, or $c^i \geq c_*^i$. Moreover, bank i is in $\mathcal{D}_N \setminus \mathcal{C}_N$ for $e^i \leq e_1^i$ or $c^i \leq c_1^i$, and bank i is in $\mathcal{S}_N \setminus \mathcal{I}_N$ for $c^i \geq c_2^i$.*

Proof. Observe first that a bank i is solvent if and only if $L^i + w^i \leq c^i + \alpha_N e^i + (\pi p_N)^i$. Since α_N and p_N do not depend on the amount of the illiquid asset held by a solvent bank, it follows that if i is solvent for some e_*^i, it is solvent for all $e^i > e_*^i$. This establishes the cut-off form of bank i's solvency. Moreover, the cutoff is finite because $\alpha_N > 0$. Since α_N and p_N are non-decreasing in c^i and non-increasing in w^i by Statement 3 of Lemma E.3, it follows that there exist a cutoff c_*^i such that bank i is solvent if and only if $c^i \geq c_*^i$.

For the second statement, observe that $i \in \mathcal{D}_N \setminus \mathcal{C}_N$ if and only if $w^i - \beta(c^i + \alpha_N e^i + (\pi p_N)^i) > 0$ and that $i \in \mathcal{S}_N \setminus \mathcal{I}_N$ if and only if $\ell_N^i = 0$. Since for a bank in $\mathcal{D}_N \setminus \mathcal{C}_N$, a change in e^i or c^i does not affect the asset recovery rate or the vector of repayments, it follows that if a bank i is in $\mathcal{D}_N \setminus \mathcal{C}_N$ for some e_1^i (some c_1^i), then it is also in $\mathcal{D}_N \setminus \mathcal{C}_N$ for $e^i \leq e_1^i$ (for $c^i \leq c_1^i$). Since $\mathcal{D}_N \setminus \mathcal{C}_N \subseteq \mathcal{D}_N$, it follows that $e_1^i \leq e_*^i$ ($c_1^i \leq c_*^i$). Similarly, for a bank in $\mathcal{S}_N \setminus \mathcal{I}_N$, a change in c^i does not affect the asset recovery rate or the vector of repayments, hence if a bank i is in $\mathcal{S}_N \setminus \mathcal{I}_N$ for some c_2^i, then it is also in $\mathcal{S}_N \setminus \mathcal{I}_N$ for $c^i \geq c_2^i$. Again, it follows that $c_*^i \leq c_2^i$ by monotonicity. Finally, as c^i goes to infinity, bank i will not have to liquidate anything, hence $i \in \mathcal{S}_N \setminus \mathcal{I}_N$, showing that c_2^i is finite. □

Lemma G.8. *For each i, let e_*^i, c_1^i, c_*^i, and c_2^i be as in Lemma G.7. Then α_N is decreasing for $e < e_*^i$, α_N has a positive discontinuity at e_*^i, and α_N is constant for $e^i > e_*^i$. The asset recovery rate α_N is constant for $c^i \leq c_1^i$ and $c^i \geq c_2^i$, increasing for $c^i \in [c_*^i, c_2^i]$, non-decreasing for $c^i \in [c_1^i, c_*^i]$ with strict monotonicity if $\theta_{\mathcal{I}_N}^i(\beta, \pi) > 0$. The asset recovery rate α_N is decreasing in γ, it is non-decreasing in β, and constant in λ. Moreover, α_N is locally increasing in β if and only if at least one bank in $\mathcal{C}_N$ has a creditor in $\mathcal{I}_N$.*

In order to prove Lemma G.8, we require the following two auxiliary results. The first implies that $\theta_{\mathcal{S}}^i(\beta, \pi) \leq 1$ for every $i \in \mathcal{C}_N$ and any set $\mathcal{S}$ by setting $y^{\mathcal{C}} = \rho_i^{\mathcal{C}_N}$.

Lemma G.9. *For any two disjoint sets of banks $\mathcal{I}$ and $\mathcal{C}$ and any vector $y \in [0,\infty)^n$, we have*

$$\sum_{i\in\mathcal{I}} \pi^{i,\mathcal{C}} \big(I - \beta\pi^{\mathcal{C},\mathcal{C}}\big)^{-1} y^{\mathcal{C}} \le \sum_{i\in\mathcal{C}} y^i.$$

Proof. Let $x^{\mathcal{C}} := \big(I - \beta\pi^{\mathcal{C},\mathcal{C}}\big)^{-1} y^{\mathcal{C}}$. We can expand $x^{\mathcal{C}}$ using a power series to $x^{\mathcal{C}} = \sum_{k=0}^{\infty} \big(\beta\pi^{\mathcal{C},\mathcal{C}}\big)^k y^{\mathcal{C}}$. Since every entry of y and π is non-negative, it follows that $x^i \ge y^i \ge 0$. Let $\mathbf{1}_{\mathcal{C}} \in \mathbb{R}^{|\mathcal{C}|}$ denote the vector with ones in every component. Then, we can write

$$\beta \sum_{i,j\in\mathcal{C}} \pi^{ij} x^j = \beta \mathbf{1}_{\mathcal{C}}^\top \pi^{\mathcal{C},\mathcal{C}} x^{\mathcal{C}} + \mathbf{1}_{\mathcal{C}}^\top x^{\mathcal{C}} - \mathbf{1}_{\mathcal{C}}^\top x^{\mathcal{C}} = \mathbf{1}_{\mathcal{C}}^\top x^{\mathcal{C}} - \mathbf{1}_{\mathcal{C}}^\top \big(I - \beta\pi^{\mathcal{C},\mathcal{C}}\big) x^{\mathcal{C}} = \sum_{i\in\mathcal{C}} (x^i - y^i). \tag{85}$$

Since $\mathcal{I}$ and $\mathcal{C}$ are disjoint and π is column-stochastic, we obtain

$$\beta \sum_{i\in\mathcal{I}} \pi^{i,\mathcal{C}} x^{\mathcal{C}} = \beta \sum_{j\in\mathcal{C}} x^j \sum_{i\in\mathcal{I}} \pi^{ij} \le \beta \sum_{j\in\mathcal{C}} x^j \left(1 - \sum_{i\in\mathcal{C}} \pi^{ij}\right) = \sum_{i\in\mathcal{C}} \big(y^i - (1-\beta) x^i\big) \le \beta \sum_{i\in\mathcal{C}} y^i, \tag{86}$$

where we have used (85) in the penultimate equation and $y^i \le x^i$ in the last equation. □

Lemma G.10. *At any differentiability point of any financial system, the quantity*

$$\chi_N := \frac{\gamma}{\alpha_N} \sum_{i\in\mathcal{I}_N} \big(L^i + w^i - c^i - (\pi p_N)^i\big) + \gamma \sum_{i\in\mathcal{I}_N} \frac{\partial (\pi p_N)^i}{\partial \alpha_N}$$

is strictly smaller than 1.

Proof. Since there is at least one fundamentally defaulting bank, it follows that $\sum_{i=1}^n \ell_p^i > 0$ for any vector of repayments p. Therefore, (48) implies that $\frac{\partial \alpha_p}{\partial \gamma} < 0$. Since α_p is non-decreasing in p and p_N is non-increasing in γ by Lemma E.3 it follows from total differentiation that

$$\frac{\partial \alpha_N}{\partial \gamma} = \frac{\partial \alpha_p}{\partial p} \frac{\partial p_N}{\partial \gamma} + \frac{\partial \alpha_p}{\partial \gamma} < 0. \tag{87}$$

Observe that we can write

$$\alpha_N = \exp\left(-\frac{\gamma}{\alpha_N} \sum_{i\in\mathcal{I}_N} \big(L^i + w^i - c^i - (\pi p_N)^i\big) - \gamma \sum_{i\in\mathcal{D}_N} e^i \right). \tag{88}$$

Implicitly differentiating (88) with respect to γ, we obtain

$$\frac{\partial \alpha_N}{\partial \gamma} = \frac{\alpha_N \ln(\alpha_N)}{\gamma} + \chi_N \frac{\partial \alpha_N}{\partial \gamma} = \frac{\alpha_N \ln(\alpha_N)}{\gamma(1-\chi_N)}. \tag{89}$$

By (87), this term has to be negative, implying that $\chi_N < 1$. □

Proof of Lemma G.8. Since λ is just a welfare parameter, it is clear that α_N is constant in λ. It follows from (87) that α_N is strictly decreasing in γ at continuity points and it follows from

Statement 3 of Lemma E.3 that α_N is decreasing in γ at discontinuities. Statement 3 of Lemma E.3 shows that α_N is non-decreasing in β.

Let us now characterize when α_N is strictly increasing in β. Fix a bank $i \in \mathcal{I}_N$ and observe that $\ell_N^i = \frac{1}{\alpha_N}(L^i + w^i - c^i - (\pi p_N)^i)$. Taking the partial derivative with respect to β yields

$$\frac{\partial \ell_N^i}{\partial \beta} = -\frac{1}{\alpha_N}\ell_N^i \frac{\partial \alpha_N}{\partial \beta} - \frac{1}{\alpha_N}\sum_{j\in\mathcal{C}_N} \pi^{ij}\frac{\partial p_N^j}{\partial \beta}, \tag{90}$$

where we have used the fact that p_N^i is constant for solvent banks and banks in $\mathcal{D}_N \setminus \mathcal{C}_N$. Since ℓ_N^j is constant for $j \notin \mathcal{I}_N$, multiplying (90) by $-\gamma\alpha_N$ and summing over all banks yields

$$\frac{\partial \alpha_N}{\partial \beta} = -\gamma\alpha_N \sum_{i\in\mathcal{I}_N} \frac{\partial \ell_N^i}{\partial \beta} = \gamma \sum_{i\in\mathcal{I}_N} \ell_N^i \frac{\partial \alpha_N}{\partial \beta} + \gamma \sum_{i\in\mathcal{I}_N}\sum_{j\in\mathcal{C}_N} \pi^{ij}\frac{\partial p_N^j}{\partial \beta} = \frac{\gamma \sum_{i\in\mathcal{I}_N}\sum_{j\in\mathcal{C}_N} \pi^{ij}\frac{\partial p_N^j}{\partial \beta}}{1-\gamma\sum_{i\in\mathcal{I}_N}\ell_N^i}.$$

By Statement 3 of Lemma E.3, this quantity is non-negative and it is different from 0 if and only if there exists a bank $j \in \mathcal{C}_N$ with a creditor $i \in \mathcal{I}_N$.

For the sensitivity with respect to e^i and c^i, we first determine the change in πp_N in α_N. Let $\mathcal{S}$ be any set of banks with $\mathcal{S} \cap \mathcal{C}_N = \emptyset$. Since p_N^j is constant for any $j \notin \mathcal{C}_N$, it follows from (84) that

$$\sum_{j\in\mathcal{S}} \frac{\partial(\pi p_N)^j}{\partial \alpha_N} = \beta \mathbf{1}_{\mathcal{S}}^\top \pi^{\mathcal{S},\mathcal{C}_N}\left(I - \beta\pi^{\mathcal{C}_N,\mathcal{C}_N}\right)^{-1} e^{\mathcal{C}_N} = \sum_{j\in\mathcal{C}_N}\left(\beta\theta^j_{\mathcal{S}\setminus\mathcal{D}_N} + \theta^j_{\mathcal{S}\cap\mathcal{D}_N\setminus\mathcal{C}_N}\right)e^j \geq 0 \tag{91}$$

at any differentiability point. In the same way, it follows that

$$\sum_{j\in\mathcal{S}} \frac{\partial(\pi p_N)^j}{\partial e^i} = \sum_{j\in\mathcal{S}} \frac{\partial(\pi p_N)^j}{\partial \alpha_N}\frac{\partial \alpha_N}{\partial e^i} + \alpha_N\left(\beta\theta^i_{\mathcal{S}\setminus\mathcal{D}_N}(\beta,\pi) + \theta^i_{\mathcal{S}\cap\mathcal{D}_N\setminus\mathcal{C}_N}(\beta,\pi)\right)1_{\{i\in\mathcal{C}_N\}}. \tag{92}$$

Using (92) for $\mathcal{S} = \mathcal{I}_N$, implicit differentiation in (88) with respect to e^i yields

$$\frac{\partial \alpha_N}{\partial e^i} = \chi_N \frac{\partial \alpha_N}{\partial e^i} - \gamma\alpha_N 1_{\{i\in\mathcal{D}_N\}} + \gamma\alpha_N\beta\theta^i_{\mathcal{I}_N}(\beta,\pi)1_{\{i\in\mathcal{C}_N\}}. \tag{93}$$

For $i \in \mathcal{C}_N$, solving (93) for $\frac{\partial \alpha_N}{\partial e^i}$ and using that $\theta^i_{\mathcal{I}_N}(\beta,\pi) \leq 1$ by Lemma G.9, we obtain

$$\frac{\partial \alpha_N}{\partial e^i} \leq \frac{\gamma\alpha_N(\beta-1)}{1-\chi_N} < 0,$$

where we have used that $\chi_N < 1$ by Lemma G.10. It follows in the same way that the partial derivative in (93) is negative for $i \in \mathcal{D}_N \setminus \mathcal{C}_N$ and 0 for $i \notin \mathcal{D}_N$. For discontinuity points, observe that Lemma G.7 shows that crossing e^i_* reduces bankruptcy costs, hence p_N and α_N have an upward discontinuity. All other discontinuities occur when an increase in e^i decreases the recovery rate to a level, at which another bank becomes insolvent, causing a downward discontinuity in p_N and α_N.

For the sensitivity with respect to c^i, observe first that at continuity points, (84) implies that the weak partial derivative of $(\pi p_N)^i$ with respect to c^i equals

$$\sum_{j\in\mathcal{I}_N}\frac{\partial(\pi p_N)^j}{\partial c^i}=\sum_{j\in\mathcal{I}_N}\frac{\partial(\pi p_N)^j}{\partial\alpha_N}\frac{\partial\alpha_N}{\partial c^i}+\beta\theta^i_{\mathcal{I}_N}(\beta,\pi)1_{\{i\in\mathcal{C}_N\}},\tag{94}$$

Using (94), implicit differentiation in (88) with respect to c^i yields

$$\frac{\partial\alpha_N}{\partial c^i}=\gamma 1_{\{i\in\mathcal{I}_N\}}+\chi_N\frac{\partial\alpha_N}{\partial c^i}+\gamma\beta\theta^i_{\mathcal{I}_N}(\beta,\pi)1_{\{i\in\mathcal{C}_N\}}.\tag{95}$$

Consider first the case where $c^i\le c^i_1$ or $c^i\ge c^i_2$. Then $i\notin\mathcal{I}_N\cup\mathcal{C}_N$ by Lemma G.7, hence it follows from (95) that α_N is constant in c^i. If $c^i\in[c^i_1,c^i_*]$, then $i\in\mathcal{I}_N$ by Lemma G.7. Solving (95) for $\frac{\alpha_N}{\partial c^i}$ shows that α_N is increasing in c^i by Lemma G.10. Finally, if $c^i\in[c^i_*,c^i_2]$, then $i\in\mathcal{C}_N$ by Lemma G.7. It follows from (95) α_N is increasing in c^i at continuity points if $\theta^i_{\mathcal{I}_N}(\beta,\pi)$ is positive and α_N is constant otherwise. Statement 3 of Lemma E.3 shows that α_N is increasing in c^i at discontinuities. The statement now follows from Lemma G.7. □

Lemma G.11. *Welfare losses W_N are increasing in γ, non-decreasing in λ, and non-increasing in β. Welfare losses W_N are locally increasing in λ if and only if $\theta^i(\beta,\pi)>0$ for some bank $i\in\mathcal{D}_N\setminus\mathcal{C}_N$ and W_N is locally decreasing in β if and only if $\theta^i(\beta,\pi)>0$ for some bank $i\in\mathcal{D}_N$.*

Proof. It follows immediately from the definition of welfare losses in (6) that W_N is non-decreasing in λ and that it is increasing if and only if $\delta^i(p_N,\alpha_N)>0$ for at least one bank $i\in\mathcal{D}_N\setminus\mathcal{C}_N$. The latter condition is equivalent to $\theta^i(\beta,\pi)$ for one bank $i\in\mathcal{D}_N\setminus\mathcal{C}_N$. For the sensitivity with respect to the remaining parameters, note that $\sum_{i=1}^n\ell^i_N=-\ln(\alpha_N)/\gamma$ and (**??**) imply that

$$W_N=\frac{(\alpha_N-1)\ln(\alpha_N)}{\gamma}+\sum_{i\in\mathcal{D}_N}\left(c^i+\alpha_Ne^i+(\pi p_N)^i-w^i-p^i_N+(1+\lambda)\delta^i(p_N,\alpha_N)\right).$$

Adding $\sum_{i\in\mathcal{S}_N}(\pi p_N)^i-p^i_N+L^i-\sum_{i\in\mathcal{S}_N}(\pi p_N)^i=0$ and using the fact that π is column stochastic, it follows there exists a constant $C>0$ that does not depend on γ,β, c, or e such that

$$W_N=C+\frac{(\alpha_N-1)\ln(\alpha_N)}{\gamma}+\sum_{i\in\mathcal{D}_N}(c^i+\alpha_Ne^i)-\sum_{i\in\mathcal{S}_N}(\pi p_N)^i-\tilde\beta\sum_{i\in\mathcal{D}_N\setminus\mathcal{C}_N}(c^i+\alpha_Ne^i+(\pi p_N)^i),\tag{96}$$

where we have abbreviated $\tilde\beta:=(1+\lambda)\beta$. As a preliminary step, let us calculate the partial derivatives of W_N with respect to α_N. It follows from (91), (96) that

$$\frac{\partial W_N}{\partial\alpha_N}=\frac{\alpha_N\ln(\alpha_N)+\alpha_N-1}{\gamma\alpha_N}+\sum_{i\in\mathcal{D}_N}(1-\beta(\theta^i+\lambda\theta^i_{\mathcal{D}_N}))e^i,\tag{97}$$

where we recall that $\theta^i_{\mathcal{D}_N}=\theta^i=1$ for $i\in\mathcal{D}_N\setminus\mathcal{C}_N$. It follows from (87), (89), (96), (97), and the fact that p_N depends on γ only through α_N that

$$\frac{\partial W_N}{\partial \gamma} = \frac{\alpha_N \ln^2(\alpha_N) - \ln(\alpha_N)(1-\alpha_N)\chi_N}{\gamma^2(1-\chi_N)} + \sum_{i\in\mathcal{D}_N} (1-\beta(\theta^i + \lambda\theta^i_{\mathcal{D}_N}))e^i \frac{\partial \alpha_N}{\partial \gamma} \tag{98}$$

$$\geq -\frac{\ln(\alpha_N)(1-\alpha_N)\chi_N}{\gamma^2(1-\chi_N)} - \beta \sum_{i\in\mathcal{D}_N} (\theta^i + \lambda\theta^i_{\mathcal{D}_N})e^i \frac{\partial \alpha_N}{\partial \gamma} > 0, \tag{99}$$

where we have used $\sum_{i\in\mathcal{D}_N} e^i \leq \sum_{i=1}^n \ell_N^i = -\ln(\alpha_N)/\gamma$ and that α_N is decreasing in γ in the second inequality. Turning to β, differentiating (3) implicitly and solving for $p_N^{\mathcal{C}_N}$ yields

$$\frac{\partial p_N^{\mathcal{C}_N}}{\partial \beta} = \frac{\partial p_N^{\mathcal{C}_N}}{\partial \alpha_N}\frac{\partial \alpha_N}{\partial \beta} + \left(I - \beta\pi^{\mathcal{C}_N,\mathcal{C}_N}\right)^{-1}\left(c^{\mathcal{C}_N} + \alpha_N e^{\mathcal{C}_N} + (\pi p_N)^{\mathcal{C}_N}\right).$$

Since the dependence of W_N on α_N through p_N is already captured in (97), it follows that the derivative of (96) with respect to β is equal to

$$\frac{\partial W_N}{\partial \beta} = \frac{\partial W_N}{\partial \alpha_N}\frac{\partial \alpha_N}{\partial \beta} - \sum_{i\in\mathcal{D}_N} (\theta^i + \lambda\theta^i_{\mathcal{D}_N})(c^i + \alpha_N e^i + (\pi p_N)^i). \tag{100}$$

It follows from (97) and $\sum_{i\in\mathcal{D}_N} e^i \leq \sum_{i=1}^n \ell_N^i = -\ln(\alpha_N)/\gamma$ that

$$\frac{\partial W_N}{\partial \alpha_N} \leq -\frac{1-\alpha_N}{\gamma\alpha_N} - \beta\sum_{i\in\mathcal{D}_N}(\theta^i + \lambda\theta^i_{\mathcal{D}_N})e^i < 0. \tag{101}$$

Together with Lemma G.8, equations (100) and (101) imply that W_N is non-increasing in β and W_N is strictly decreasing in β if and only if at least one bank $j \in \mathcal{C}_N$ has a creditor $i \notin \mathcal{C}_N$ or if $\mathcal{D}_N \setminus \mathcal{C}_N$ is non-empty. The former condition is equivalent to $\theta^i(\beta,\pi) > 0$ for at least one $i \in \mathcal{C}_N$ and the latter condition is equivalent to $\theta^i(\beta,\pi) > 0$ for at least one $i \in \mathcal{D}_N \setminus \mathcal{C}_N$. □

Lemma G.12. *For each bank i, let e_1^i and e_*^i be as in Lemma G.7. Welfare losses W_N are increasing for $e^i \leq e_1^*$ and constant for $e^i \geq e_*^i$ with a downward discontinuity at e_*^i. On the interval (e_1^i, e_*^i), welfare losses W_N have only upward discontinuities and at continuity points, W_N is locally increasing if and only if*

$$\frac{\beta(\theta^i + \lambda\theta^i_{\mathcal{D}_N}) - 1}{1 - \beta\theta^i_{\mathcal{I}_N}} \leq \frac{1-\alpha_N(1-\chi_N)}{1-\chi_N} + \gamma\alpha_N \sum_{j\in\mathcal{D}_N} \frac{\beta(\theta^j_{\mathcal{I}_N^c} + \lambda\theta^j_{\mathcal{D}_N})}{1-\chi_N} e^j. \tag{102}$$

Observe that the right-hand side of (102) is positive. Therefore, if $\lambda \leq \frac{1-\beta}{\beta}$, then (102) is satisfied and W_N is increasing for all $e^i < e_*^i$ and any bank i. Moreover, if $\theta^i_{\mathcal{D}_N} = 0$, that is, bank i is not liable to any banks that renege on their junior creditors, then (102) is satisfied and W_N is increasing for all $e^i < e_*^i$. This is the case, in particular, if $\mathcal{D}_N \setminus \mathcal{C}_N = \emptyset$. Finally, the left-hand side of (102) is maximal if i is liable only to banks in $\mathcal{D}_N \setminus \mathcal{C}_N$. Then the left-hand side equals $\beta(1+\lambda)-1$ and W_N may be locally decreasing for $e^i \in [e_1^i, e_*^i)$: while liquidation losses and bankruptcy costs increase in e^i, losses δ^j of the senior creditors of any of i's creditors j are reduced. If the regulator values losses of senior creditors very highly (λ high), this may cause an overall increase in welfare.

Proof. If $e^i > e^i_*$, then $i \in \mathcal{S}_N$ by Lemma G.7, hence α_N and p_N are constant in e^i by Lemma G.8. Therefore, W_N is constant in e^i for $e^i > e^i_*$. Suppose, therefore, that $e^i < e^i_*$ so that $i \in \mathcal{D}_N$ by Lemma G.7. It follows from (93) and (97) that W_N depends on e^i though α_N via

$$\begin{aligned}\frac{\partial W_N}{\partial \alpha_N}\frac{\partial \alpha_N}{\partial e^i} &= \left(\frac{1-\alpha_N-\alpha_N\ln(\alpha_N)}{1-\chi_N} + \frac{\gamma\alpha_N}{1-\chi_N}\sum_{j\in\mathcal{D}_N}(\beta(\theta^j+\lambda\theta^j_{\mathcal{D}_N})-1)e^j\right)(1-\beta\theta^i_{\mathcal{I}_N}1_{\{i\in\mathcal{C}_N\}})\\ &= \left(\frac{1-\alpha_N(1-\chi_N)}{1-\chi_N} + \gamma\alpha_N\sum_{j\in\mathcal{D}_N}\frac{\beta(\theta^j_{\mathcal{I}^c_N}+\lambda\theta^j_{\mathcal{D}_N})}{1-\chi_N}e^j\right)(1-\beta\theta^i_{\mathcal{I}_N}1_{\{i\in\mathcal{C}_N\}}),\end{aligned} \tag{103}$$

where we have used

$$1-\chi_N = 1+\ln(\alpha_N)+\gamma\sum_{j\in\mathcal{D}_N}(1-\theta^j_{\mathcal{I}_N})e^j$$

in the second equation of (103). As in the proof of Lemma G.11, the partial derivative of $p^{\mathcal{C}_N}_N$ is

$$\frac{\partial p^{\mathcal{C}_N}_N}{\partial e^i} = \frac{\partial p^{\mathcal{C}_N}_N}{\partial \alpha_N}\frac{\partial \alpha_N}{\partial e^i} + (I-\beta\pi^{\mathcal{C}_N,\mathcal{C}_N})^{-1}\beta\alpha_N\rho^{\mathcal{C}_N}_i 1_{\{i\in\mathcal{C}_N\}}.$$

Since the dependence of W_N on α_N through p_N is already captured in (97), it follows that the derivative of (96) with respect to e^i is equal to

$$\begin{aligned}\frac{\partial W_N}{\partial e^i} &= \frac{\partial W_N}{\partial \alpha_N}\frac{\partial \alpha_N}{\partial e^i} + \alpha_N - \sum_{j\in\mathcal{C}^c_N}\pi^{\{j\},\mathcal{C}_N}\left(\frac{\partial p^{\mathcal{C}_N}_N}{\partial e^i} - \frac{\partial p^{\mathcal{C}_N}_N}{\partial \alpha_N}\frac{\partial \alpha_N}{\partial e^i}\right)\\ &= \frac{\partial W_N}{\partial \alpha_N}\frac{\partial \alpha_N}{\partial e^i} + \alpha_N - (\theta^i+\lambda\theta^i_{\mathcal{D}_N})\beta\alpha_N 1_{\{i\in\mathcal{C}_N\}}.\end{aligned} \tag{104}$$

If $e^i \le e^i_1$, then $i \in \mathcal{D}_N \setminus \mathcal{C}_N$ by Lemma G.7. Thus, (104) is simply the sum of α_N and (103), hence positive. If $e^i \in (e^i_1, e^i_*)$, then $i \in \mathcal{C}_N$, hence (102) follows immediately from (103) and (104). At $e^i = e^i_*$, it follows from Lemma G.7 that bank j becomes solvent and hence W_N has a downward discontinuity. As in the proof of Lemma G.8, an increase in e^i at $e^i \neq e^i_*$ can lead to changes in $\mathcal{D}_N$ only through other banks defaulting. At those points, there are upward discontinuities in W_N. □

Lemma G.13. *Let $0 \le c^i_1 \le c^i_* \le c^i_2$ be as in Lemma G.7. Welfare losses W_N are continuously increasing in c^i for $c^i \le c^i_1$ if and only if $(1+\lambda)\beta < 1$, W_N is decreasing for $c^i \in [c^i_*, c^i_2]$ with a downward discontinuity at c^i_*, and constant for $c^i \ge c^i_2$. On the interval $[c^i_1, c^i_*]$, welfare losses W_N have only downward discontinuities and they are locally decreasing in c^i if and only if*

$$\frac{1-\beta(\theta^i+\lambda\theta^i_{\mathcal{D}_N})}{\beta\theta^i_{\mathcal{I}_N}} \le \frac{1-\alpha_N(1-\chi_N)}{\alpha_N(1-\chi_N)} + \gamma\sum_{j\in\mathcal{D}_N}\frac{\beta(\theta^j_{\mathcal{I}^c_N}+\lambda\theta^j_{\mathcal{D}_N})}{1-\chi_N}e^j. \tag{105}$$

Proof. Observe first that α_N and p_N are constant in c^i for $c^i \notin [c_1^i, c_2^i]$ by Lemma G.8 and (94). For $c^i \leq c_1^i$, Lemma G.7 shows that $i \in \mathcal{D}_N \setminus \mathcal{C}_N$. Thus, it follows straight from (96) that W_N is increasing in c^i if and only if $\tilde{\beta} < 1$. Similarly, for $c^i \geq c_2^i$, Lemma G.7 states that $i \in \mathcal{S}_N \setminus \mathcal{I}_N$, hence W_N is locally constant by (96). Suppose now that $c^i \in [c_1^i, c_2^i]$. Because p and α are non-decreasing in c^i by Lemma E.3, an increase in c^i can affect solvency of other banks only by making them solvent, leading to downward discontinuities in W_N. At continuity points, welfare losses are weakly differentiable. In the same way as in (103), it follows from (95) and (97) that

$$\frac{\partial W_N}{\partial \alpha_N}\frac{\partial \alpha_N}{\partial c^i} = -\left(\frac{1-\alpha_N(1-\chi_N)}{\alpha_N(1-\chi_N)} + \gamma \sum_{j \in \mathcal{D}_N} \frac{\beta(\theta^j_{\mathcal{I}_N^c} + \lambda\theta^j_{\mathcal{D}_N})}{1-\chi_N} e^j\right)\left(1_{\{c^i \in [c_*^i, c_2^i)\}} + \beta\theta^i_{\mathcal{I}_N} 1_{\{c^i < c_*^i\}}\right). \quad (106)$$

It follows from (84) that

$$\frac{\partial p_N^{\mathcal{C}_N}}{\partial c^i} = \frac{\partial p_N^{\mathcal{C}_N}}{\partial \alpha_N}\frac{\partial \alpha_N}{\partial c^i} + \left(I - \beta\pi^{\mathcal{C}_N,\mathcal{C}_N}\right)^{-1}\beta\rho_i^{\mathcal{C}_N} 1_{\{c^i < c_*^i\}},$$

where we have used that $i \in \mathcal{C}_N$ if and only if $c^i \in [c_1^i, c_*^i)$ by Lemma G.7. We conclude that

$$\frac{\partial W_N}{\partial c^i} = \frac{\partial W_N}{\partial \alpha_N}\frac{\partial \alpha_N}{\partial c^i} + 1_{\{c^i < c_*^i\}} - \beta(\theta^i + \lambda\theta^i_{\mathcal{D}_N})1_{\{c^i < c_*^i\}}. \quad (107)$$

If $c^i \in [c_*^i, c_2^i]$, then it follows straight from (107) that W_N is decreasing in c^i. If $c^i < c_*^i$, Since α_N is increasing in c^i by Lemma G.8, it follows from (107) that W_N is decreasing in c^i. For $c^i \in [c_1^i, c_*^i)$, (105) follows from (106) and (107). □

G.3 Credibility of the Threat

Proof of Lemma 4.3. Observe first that welfare losses W_P in the public bailout do not depend on β. The desired statement thus follows straight from Lemma G.11. For the statement with respect to γ, continuity of W_P and Lemma G.11 imply that all discontinuities are downward discontinuities. Since W_P depends neither on β or on θ^i, the second statement follows straight from (98) after observing that α_N is decreasing in γ and that χ_N is increasing in β and θ^i.

For the sensitivity with respect to λ, observe that the marginal decrease of $\sum_{j \in \mathcal{D}_N(s)} \delta^j(s)$ with respect to the provided subsidy s^i is $\theta^i_{\mathcal{D}_N \setminus \mathcal{C}_N}(\beta, \pi) \leq 1$. For $s = s_L$, no bank defaults and hence $\delta(s_L) = 0$. It follows that

$$\frac{\partial W_N}{\partial \lambda} = \sum_{j \in \mathcal{D}_N \setminus \mathcal{C}_N} \delta^j(p_N, \alpha_N) < \sum_{i=1}^n s_L^i \leq \sum_{i=1}^n s_0^i + \frac{\alpha_P \ln(\alpha_P)}{\gamma} = \frac{\partial W_P}{\partial \lambda},$$

where the first inequality is strict since $e^i > 0$ for every bank i by assumption and we have used (79) in the last equality.

For the sensitivity with respect to e^i, observe that W_P is continuous with respect to e^i and, by Lemma G.11, W_N has downward discontinuities except at e_*^i. Therefore, the only upward discontinuity of $W_P - W_N$ occurs at e_*^i. At continuity points, $W_P - W_N$ is weakly differentiable.

Lemmas G.5 and G.12 imply together that $W_P - W_N$ is decreasing if (102) is satisfied. $\square$

Proof of Lemma 4.4. We first show monotonicity of $W_P - W_N$ in c^i. Monotonicity in ε^i then follows from $c^i = c_0^i - \varepsilon^i$ via the chain rule. Recall first that W_P is continuous and W_N has only downward discontinuities by Lemma G.13. Therefore, $W_P - W_N$ has only upward discontinuities. We deduce in the same way as in the proof of Lemma G.7 that there exists $0 \le c_a^i \le c_b^i$ such that i does not have to liquidate anything in the clearing equilibrium (L, ℓ_L, α_L) if and only if $c^i \ge c_b^i$ and that $i \in \mathcal{F}$ if and only if $c^i < c_a^i$. Since interbank claims are valued higher in (L, ℓ_L, α_L) than in (p_N, ℓ_N, α_N), more cash is required in the latter clearing equilibrium to reach solvency and liquidity, that is, $c_2^i \ge c_b^i$ and $c_*^i \ge c_a^i$, where c_2^i and c_*^i are defined in Lemma G.7. By definition of c_b^i we can write $s_0^i = (c_b^i - c^i)^+$ and hence $-\frac{\partial s_0^i}{\partial w^i} = 1_{\{c^i \le c_b^i\}}$. Since the complete bailout depends on c^i only through s_0^i it follows that W_P and α_P are constant in c^i for $c^i \ge c_b^i$ and that $\frac{\partial \alpha_P}{\partial c^i} = -\frac{\partial \alpha_P}{\partial s_0^i}$ and $\frac{\partial W_P}{\partial c^i} = -\frac{\partial W_P}{\partial s_0^i} \ge -\lambda$ otherwise. It follows from (83) and Lemma G.1 that

$$\frac{\partial W_P}{\partial c^i} = -\lambda(1 - g'(\alpha_P)\xi 1_{\{c^i > c_a^i\}})1_{\{c^i \le c_b^i\}}. \tag{108}$$

Since W_N is decreasing for $c^i \in [c_*^i, c_2^i]$ and constant for $c^i \ge c_2^i$ by Lemma G.13, it follows immediately from (108) that $W_P - W_N$ is increasing for $c^i \in [\max(c_*^i, c_b^i), c_2^i]$ and constant for $c^i \ge c_2^i$. Since $\frac{\partial W_N}{\partial c^i} = 1 - (1+\lambda)\beta$ for $c^i \le c_1^i$, it follows from (108) that $\frac{\partial (W_P - W_N)}{\partial c^i} = -(1+\lambda)(1-\beta) < 0$ for $c^i \in [0, \min(c_1^i, c_a^i)]$. For $c^i \in [c_1^i, c_b^i]$, the partial derivative of W_N with respect to c^i is given by (107). In particular, $\frac{\partial W_N}{\partial c^i}$ is decreasing in β and $\theta^j(\beta, \pi)$ for any $j \in \mathcal{C}_N$. For $c^i \in [c_a^i, c_1^i]$, the partial derivative of W_N is given by $1 - (1+\lambda)\beta$, which is also decreasing in β and trivially non-increasing in $\theta^i(\beta, \pi)$. Since $\frac{\partial W_P}{\partial c^i}$ is constant in β and $\theta^j(\beta, \pi)$, it follows that $\frac{\partial (W_P - W_N)}{\partial c^i}$ is increasing in β and $\theta^j(\beta, \pi)$ for any $j \in \mathcal{C}_N$ and decreasing in λ. The result thus follows from the chain rule and by setting $\varepsilon_1^i := c_0^i - c_2^i$, $\varepsilon_2^i := c_0^i - \max(c_b^i, c_*^i)$, and $\varepsilon_3^i := c_0^i - \max(c_1^i, c_a^i)$. $\square$

Proof of Lemma 4.2. It follows as an application of Lemma G.9 to $y^{\mathcal{C}} = \rho_i^{\mathcal{C}_N}$ that $\theta^i(\beta, \pi) \in [0, 1]$ for any bank i. Monotonicity in β follows immediately from $(I - \beta \pi^{\mathcal{C}_N, \mathcal{C}_N})^{-1} = \sum_{k=0}^{\infty} (\beta \pi^{\mathcal{C}_N, \mathcal{C}_N})^k$. For the second statement, consider $\mathcal{D} = \mathcal{D}_N$, $\mathcal{C} = \mathcal{C}_N$, $\mathcal{S} = \mathcal{D}_N^c$, and $\mathcal{I} = \mathcal{I}_N$ as fixed. Observe from (88) that the recovery rate α_N depends on p_N only through πp_N. It follows from (84) that

$$(\pi p_N)^i = \sum_{j \in \mathcal{C}} \theta_i^j(\beta, \pi) \left(\beta c^j + \beta \alpha_N e^j + \beta \sum_{k \in \mathcal{S}} L^{jk} - w^j \right) + \sum_{j \in \mathcal{S}} L^{ij}$$

for any bank $i \notin \mathcal{C}_N$, showing that $(\pi p_N)^i$ depends on π only through $\sum_{j \in \mathcal{C}} \theta_i^j(\beta, \pi)$ and $\sum_{j \in \mathcal{S}} L^{ij}$. Therefore, (88) implies that α_N depends on π only through $\sum_{j \in \mathcal{C}} \theta_{\mathcal{I}}^j(\beta, \pi)$ and $\sum_{j \in \mathcal{S}} L^{ij}$. Finally, it follows as a consequence to (28) that W_N depends on π only through $\sum_{j \in \mathcal{C}} \theta_{\mathcal{I}}^j(\beta, \pi)$, the total throughput $\sum_{j \in \mathcal{C}} \theta^j(\beta, \pi)$, and $\sum_{j \in \mathcal{S}} L^{ij}$. Since the complete bailout does not depend on the network structure at all, the claim follows. $\square$

H Proofs of Section 5

Proof of Lemma 5.1. Let s be any set of subsidies that rescues banks in $\mathcal{B}$ by awarding direct subsidies only to banks in $\mathcal{B}$. For any such vector of subsidies, the greatest clearing payment vector is given by $\bar{p}(s) = p(\mathcal{B})$. Indeed, for any such vector of subsidies, banks in $\mathcal{B}$ are able to repay their liabilities in full. Contingent on full repayment by banks in $\mathcal{B}$, banks in $\mathcal{B}^c$ are not affected by the subsidies as they are awarded only to banks in $\mathcal{B}$, hence $\bar{p}(s) = p(\mathcal{B})$. The welfare-maximizing vector of direct subsidies that banks $i \in \mathcal{B}$ require to afford payment L^i is thus given by the shortfall $\left(L^i + w^i - c^i - (\pi p(\mathcal{B}))^i\right)^+$. This shows that any welfare-maximizing bailout is of the form $s(\mathcal{B})$ for a suitable set $\mathcal{B}$, hence it is in $\mathcal{S}_P$. □

Proof of Lemma 5.3. Fix a feasible proposal (b, s) with accepting equilibrium response a. Necessity of Condition 1 follows in the same way as in the proof of Lemma 3.5. Therefore, the regulator's best response after rejection by any bank i in $\mathcal{C} := \left\{i \mid b^i > 0, a^i = 1\right\}$ is a public bailout. Let $\mathcal{B}_i \in \mathcal{B}_P$ be such that $r(b, s, (0, a^{-i})) = s(\mathcal{B}_i)$. Suppose now that $a^i = 1$ but Condition 2 is violated for bank i with threats from $s(\mathcal{B}_i)$. The negation of Condition 2 implies that

$$L^i + w^i \le c^i + s^i - b^i + \sum_{j=1}^{n} \pi^{ij} L^j < c^i + s^i(\mathcal{B}_*) + \sum_{j=1}^{n} \pi^{ij} p^j(\mathcal{B}_i). \tag{109}$$

This shows that bank i is solvent in the optimal bailout $s(\mathcal{B}_i)$. Therefore, subtracting $L^i + w^i$ in (109) implies that $V^i(s(\mathcal{B}_i)) > V^i(b, s, (1, a^{-i}))$. This is a contradiction. Sufficiency of the conditions follows analogously to the proof of Lemma 3.5.

For the second statement, suppose that there exists $\mathcal{B}_* \in \mathcal{B}_P$ such that Condition 2 holds in the accepting equilibrium a for every bank $i \in \mathcal{C}$ when the bailout $s(\mathcal{B}_*)$ is threatened. Define the regulator's reaction $\tilde{r}$ by setting $\tilde{r}(b, s, \tilde{a}) =$ "bailout $\mathcal{B}_*$" for any $\tilde{a}$ with $W_\lambda(b, s, \tilde{a}) > W_P^*$ and $\tilde{r}(b, s, \tilde{a}) =$ "bail-in" otherwise. Reaction $\tilde{r}$ is an equilibrium reaction that leads to the public bailout $s(\mathcal{B}_*)$ in all rejecting equilibria. Condition 2 implies that $V^i(b, s, a) \ge V^i(s(\mathcal{B}_*))$ for every bank $i \in \mathcal{C}$. Since a is an accepting equilibrium, by definition $W_\lambda(b, s, a) \le W_P^*$, hence a rejecting equilibrium is subgame Pareto efficient only if (b, s) is equivalent to some public bailout of Lemma 5.1.

For the converse, fix a rejecting equilibrium $\tilde{a}$. Let $\mathcal{B}_* \in \mathcal{B}_P$ be an arbitrary bailout that is implemented in reaction to $\tilde{a}$. Let a be any accepting equilibrium with contributing banks $\mathcal{C}$. By assumption, there exists $i \in \mathcal{C}$, for which Condition 2 is violated for $\mathcal{B}_i = \mathcal{B}_*$. Thus, bank i is strictly worse off in a than in $\tilde{a}$, hence $\tilde{a}$ cannot be Pareto dominated by an accepting equilibrium. □

H.1 Proof of Theorem 5.4

In addition to proving Theorem 5.4, we also characterize the optimal bail-in proposals.

Proposition H.1. *For any $\mathcal{B}_* \in \mathcal{B}_P$, let $\Xi_{\mathcal{B}_*}(\mathcal{B})$ denote the set of bail-in proposals (b, s) such that:*

1. *if $W_{\mathcal{B}_*}(\mathcal{B}) = W_\lambda(s(\mathcal{B})) - \sum_{j=1}^{m_{\mathcal{B}_*}(\mathcal{B})} \nu^{i^j_{\mathcal{B}_*}(\mathcal{B})}(\mathcal{B})$, then*

(a) $s = s(\mathcal{B})$

(b) $b^j = \eta^j_{\mathcal{B}_*}(\mathcal{B})$ for $j = i^1_{\mathcal{B}_*}(\mathcal{B}), \ldots, i^{m_{\mathcal{B}_*}(\mathcal{B})}_{\mathcal{B}_*}(\mathcal{B})$ and $b^j = 0$ otherwise.

2. If $W_{\mathcal{B}_*}(\mathcal{B}) = W^*_P - \lambda \eta^{i_{m_{\mathcal{B}_*}(\mathcal{B})+1}(\mathcal{B})}(\mathcal{B})$, then

(a) $s \geq s(\mathcal{B})$,

(b) $\eta^j_{\mathcal{B}_*}(\mathcal{B}) \geq b^j - s^j \geq \eta^{i_{m_{\mathcal{B}_*}(\mathcal{B})+1}}_{\mathcal{B}_*}(\mathcal{B})$ for every bank j with $b^j > 0$, and

(c) $\sum_{i=1}^n (s^i - s^i(\mathcal{B})) = \sum_{j=1}^{m_{\mathcal{B}_*}(\mathcal{B})} \eta^{i^j_{\mathcal{B}_*}(\mathcal{B})}_{\mathcal{B}_*}(\mathcal{B}) + \frac{1}{\lambda}(W^*_P - W_\lambda(s(\mathcal{B})))$.

In any subgame Pareto efficient equilibrium with welfare losses $W_{\mathcal{B}_*}(\mathcal{B})$, a bail-in from $\Xi_{\mathcal{B}_*}(\mathcal{B})$ is proposed and accepted by all banks.

We proceed to prove Theorem 5.4 and Proposition H.1 in parallel. We start by giving two precursory lemmas, which will be invoked in their proof.

Lemma H.2. *Any $(b, s) \in \Xi_{\mathcal{B}_*}(\mathcal{B})$ admits a unique accepting continuation equilibrium a with $a^i = 1$ for every bank i and $W_\lambda(b, s, a) = W_{\mathcal{B}_*}(\mathcal{B})$.*

Proof. Fix $\mathcal{B}_* \in \mathcal{B}_P$ and a set of banks $\mathcal{B}$ to be rescued. For the sake of brevity, denote $\eta = \eta_{\mathcal{B}_*}(\mathcal{B})$, $i_j = i^j_{\mathcal{B}_*}(\mathcal{B})$, and $m = m_{\mathcal{B}_*}(\mathcal{B})$. Consider a bail-in proposal $(b, s) \in \Xi_{\mathcal{B}_*}(\mathcal{B})$ with response vector $a = (1, \ldots, 1)$. Suppose first that $W_{\mathcal{B}_*}(\mathcal{B}) = W_\lambda(s(\mathcal{B})) - \lambda \sum_{j=1}^m \eta^{i_j}(\mathcal{B})$. Then the definition of $\Xi_{\mathcal{B}_*}(\mathcal{B})$ in Proposition H.1 imposes that $s = s(\mathcal{B})$ and $b^j = \eta^j$ for any bank $j \in \{i_1, \ldots, i_m\}$. It follows straight from (7) that $W_\lambda(b, s, a) = W_{\mathcal{B}_*}(\mathcal{B})$. Suppose, therefore, that $W_{\mathcal{B}_*}(\mathcal{B}) = W^*_P - \lambda \eta^{i_{m+1}}(\mathcal{B})$ instead. Then we obtain

$$W_\lambda(b, s, a) = W_\lambda(s(\mathcal{B})) + \lambda \sum_{i=1}^n (s^i - s^i(\mathcal{B})) - \lambda \sum_{j=1}^{m+1} \eta^{i_j} = W^*_P - \lambda \eta^{i_{m+1}},$$

where we have used Property 2.(c) of a bail-in in $\Xi_{\mathcal{B}_*}(\mathcal{B})$ in the last equation.

To see that a is indeed a continuation equilibrium, we verify the necessary and sufficient conditions given in Lemma 5.3. By definition of $\Xi_{\mathcal{B}_*}(\mathcal{B})$, any bank i's net contribution $b^i - s^i$ is smaller than its maximal incentive-compatible contribution $\eta^i_{\mathcal{B}_*}(\mathcal{B})$ given in (23). Therefore, the second condition in Lemma 5.3 is satisfied for every bank. For the first condition, we distinguish again the two cases. If $W_{\mathcal{B}_*}(\mathcal{B}) = W_\lambda(s(\mathcal{B})) - \lambda \sum_{j=1}^m \eta^{i_j}$, then $b^j = \eta^j$ for $j \in \{i_1, \ldots, i_m\}$. Thus, a deviation by bank i_k for $k = 1, \ldots, m$ would lead to welfare losses of

$$W_\lambda(b, s, (0, a^{-i_k})) = W_\lambda(b, s, a) + \lambda \eta^{i_k} \geq W_\lambda(b, s, a) + \lambda \eta^{i_m} \geq W^*_P,$$

where the first inequality holds because $i_1, \ldots, i_m$ is a decreasing order of the components of η and the second inequality holds by definition of m given in Theorem 5.4. This shows that Condition 1 in Lemma 5.3 is satisfied and hence a is indeed an equilibrium response. It also implies that the regulator will not agree to proceed with the bail-in if only a subset of banks accepts, thereby showing uniqueness. If $W_{\mathcal{B}_*}(\mathcal{B}) = W_P^* - \lambda\eta^{i_{m+1}}(\mathcal{B})$ holds instead, then $b^i \geq b^i - s^i \geq \eta^{i_{m+1}}$ by Property 2.(b), hence

$$W_\lambda(b, s, (0, a^{-i_k})) = W_\lambda(b, s, a) + \lambda\eta^{i_k} \geq W_\lambda(b, s, a) + \lambda\eta^{i_{m+1}} = W_P^*.$$

The remainder of the argument works analogously. □

Lemma H.3. *Let (b, s) be a feasible bail-in with equilibrium response a when threats come from the bailout $s(\mathcal{B}_*)$ for $\mathcal{B}_* \in \mathcal{B}_P$. Let $\mathcal{B}$ denote the set of solvent banks in (b, s, a) with $s^i > 0$. Then $W_\lambda(b, s, a) \geq W_{\mathcal{B}_*}(\mathcal{B})$, where the inequality is binding only if $(b, s) \in \Xi_{\mathcal{B}_*}(\mathcal{B})$.*

Proof. Fix $\mathcal{B}_* \in \mathcal{B}_P$. Let (b, s) be a bail-in with continuation equilibrium a if threats come from the bailout $s(\mathcal{B}_*)$. Let $\mathcal{B} := \{i \mid s^i > 0, \bar{p}^i(b, s, a) = L^i\}$ and abbreviate $\eta = \eta_{\mathcal{B}_*}(\mathcal{B})$ and $m = m_{\mathcal{B}_*}(\mathcal{B})$. If a is a rejecting equilibrium, then $W_\lambda(b, s, a) = W_P^*$. Thus, it follows straight from the definitions of m and $W_{\mathcal{B}_*}(\mathcal{B})$ in Theorem 5.4 that welfare losses are bounded from below by $W_{\mathcal{B}_*}(\mathcal{B})$. Suppose, therefore, that a is an accepting continuation equilibrium. Due to Lemma C.1, we may assume without loss of generality that $b^i s^i = 0$ and $a^i = 1$ for any bank i by passing to an equivalent equilibrium. Condition 1 of Lemma 5.3 implies that

$$W_\lambda(b, s, a) \geq W_P^* - \lambda b^i \tag{110}$$

for any contributing bank i. Condition 2 of Lemma 5.3 together with feasibility implies that $b^i \leq \eta^i$. Moreover, since $s(\mathcal{B})$ are the minimal subsidies that guarantee solvency of banks in $\mathcal{B}$, we must have

$$\sum_{i=1}^{n} s^i \geq \sum_{i=1}^{n} s^i(\mathcal{B}). \tag{111}$$

Consider first the case, in which there are $k < m$ contributing banks. It follows from (111) that

$$W_\lambda(b, s, a) \geq W_\lambda(s(\mathcal{B})) - \lambda \sum_{k=1}^{m-1} \eta^{i_k}(\mathcal{B}) > W_P^*,$$

where we have used in the last inequality that the sum of incentive-compatible contributions by $k \leq m - 1$ banks must be smaller than the contributions of $i_1, \ldots, i_{m-1}$ in η. This contradicts the fact that a is an accepting equilibrium. Suppose next that $k = m$. Again, the sum of incentive-compatible contributions by m banks must be smaller than the contributions of $i_1, \ldots, i_{m-1}$ in η. Together with (111), this implies

$$W_\lambda(b, s, a) \geq W_\lambda(s(\mathcal{B})) - \lambda \sum_{j=1}^{m} \eta^{i_j} \geq W_{\mathcal{B}_*}(\mathcal{B}), \tag{112}$$

where the last inequality follows directly from (24). Finally, suppose that the set of contributing banks $\mathcal{C}$ is of size $k \geq m+1$. Because $i_1, i_2, \ldots$ is a non-increasing ordering of the incentive-compatible contributions, it follows that there must be a contributing bank with $b^i \leq \eta^{i_{m+1}}$. It thus follows from (110) that

$$W_\lambda(b,s,a) \geq W_P^* - \min_{i\in\mathcal{C}} \lambda b^i \geq W_P^* - \lambda\eta^{i_{m+1}} \geq W_{\mathcal{B}_*}(\mathcal{B}). \tag{113}$$

This concludes the proof that no bail-in can attain equilibrium welfare losses below $W_{\mathcal{B}_*}(\mathcal{B})$. Lemma H.2 shows that welfare losses can be attained by bail-in proposals in $\Xi_{\mathcal{B}_*}(\mathcal{B})$. Finally, (112) and (113) imply that these inequalities hold with equality if and only if $(b,s) \in \Xi_{\mathcal{B}_*}(\mathcal{B})$. □

Proof of Theorem 5.4 and Proposition H.1. Lemma 5.3 shows that in order to rule out rejecting equilibria, the regulator must threaten the same optimal bailout to all banks. Fix now any $\mathcal{B}_*$ in $\arg\min W_\lambda(s(\mathcal{B}))$ and any $\mathcal{B} \subseteq \{1,\ldots,n\}$. Lemma H.2 shows that any $(b,s) \in \Xi_{\mathcal{B}_*}(\mathcal{B})$ has a unique accepting equilibrium with welfare losses equal to $W_{\mathcal{B}_*}(\mathcal{B})$. By Lemma 5.3, that accepting equilibrium is the unique subgame Pareto efficient continuation equilibrium. The regulator is thus aware that any bail-in from $\Xi_{\mathcal{B}_*}(\mathcal{B})$ that he proposes will be implemented in equilibrium. Moreover, by Lemma H.3, an accepted bail-in proposal from $\Xi_{\mathcal{B}_*}(\mathcal{B})$ is strictly preferred by the regulator over any other accepted proposal (b', s') that rescues banks in $\mathcal{B}$ by awarding direct subsidies only to banks in $\mathcal{B}$ when $s(\mathcal{B}_*)$ is threatened. Therefore, if $\min_{\mathcal{B}_*,\mathcal{B}} W_{\mathcal{B}_*}(\mathcal{B}) \leq W_P^*$, the regulator's only rational choice in stage 1 is to propose a bail-in from Proposition H.1. If, on the other hand, $\min_{\mathcal{B}_*,\mathcal{B}} W_{\mathcal{B}_*}(\mathcal{B}) > W_P^*$, then the regulator has no choice but to implement an optimal public bailout from Lemma 5.1. □

H.2 Properties of Optimal Bailout/Bail-in

Proof of Lemma 5.2. Fix two sets of banks $\mathcal{B}' \subseteq \mathcal{B}$. We will compare welfare losses between the bailouts $s(\mathcal{B})$ and $s(\mathcal{B}')$. Since $\gamma = 0$, the welfare losses consist only of bankruptcy costs as well as welfare costs of taxpayer contributions and senior creditor's losses. We start by comparing the subsidies. For $\mathcal{C} \in \{\mathcal{B}, \mathcal{B}'\}$, let $y(\mathcal{C}) := L + w - c - \pi p(\mathcal{C})$ so that $S(\mathcal{C}) = y(\mathcal{C})^+$ and $C(\mathcal{C}) = y(\mathcal{C})^-$, where S and C are defined in (21). Since $y(\mathcal{B}) = y(\mathcal{B}') - \zeta$, it follows that for any $i \in \mathcal{S}(\mathcal{B})$, we have

$$s^i(\mathcal{B}) = S^i(\mathcal{B}) = y^i(\mathcal{B}) + C^i(\mathcal{B}) = S^i(\mathcal{B}') - \zeta + C^i(\mathcal{B}) - C^i(\mathcal{B}'). \tag{114}$$

Since $\mathcal{B}' \subseteq \mathcal{B}$ and $p(\,\cdot\,)$ is monotonic in the set of banks $\mathcal{B}$, it follows that $\zeta^i \geq 0$ for every bank i. Therefore, $C^i(\mathcal{B}') = (-y^i(\mathcal{B}) - \zeta^i)^+ = (C^i(\mathcal{B}) - \zeta^i)^+$, where we have used that $-y^i(\mathcal{B}) - \zeta^i > 0$ only if $y^i(\mathcal{B}) < 0$. It follows from the elementary identity $a - (a-b)^+ = \min(a,b)$ that $C^i(\mathcal{B}) - C^i(\mathcal{B}') = \min(\zeta^i, C^i(\mathcal{B}))$. Together with (114), this implies that

$$\sum_{i\in\mathcal{S}(\mathcal{B})} (s^i(\mathcal{B}') - s^i(\mathcal{B})) = \sum_{i\in\mathcal{S}(\mathcal{B})} \left(\zeta^i - \min\Big(\zeta^i, C^i(\mathcal{B})\Big)\right) - \sum_{i\in\mathcal{R}} S^i(\mathcal{B}'), \tag{115}$$

where we have used that $s^i(\mathcal{B}') = S^i(\mathcal{B}')$ for $i \in \mathcal{S}(\mathcal{B}')$ and $s^i(\mathcal{B}') = 0$ for $i \in \mathcal{B} \setminus \mathcal{B}'$. For the losses of senior creditors, note that $\delta^i(\mathcal{B}) = \big(\delta^i(\mathcal{B}') - \beta\zeta^i\big)^+$ follows straight from (5). Therefore, the elementary identity $a-(a-b)^{+}=\min(a,b)$ implies

$$\delta^i(\mathcal{B}')-\delta^i(\mathcal{B})=\min\Big(\beta\zeta^i,\delta^i(\mathcal{B}')\Big). \tag{116}$$

For any set of banks $\mathcal{C}$, let $g^i(\mathcal{C}):=(1-\beta)\big(c^i+\alpha e^i+(\pi p(\mathcal{C}))^i\big)$ denote the bankruptcy losses of a bank $i\in\mathcal{D}(\mathcal{C})$. Observe that (**??**) implies

$$g^i(\mathcal{C})=c^i-w^i+(\pi p(\mathcal{C}))^i-p(\mathcal{C})^i+\delta^i(\mathcal{C})=L^i-p^i(\mathcal{C})+\delta^i(\mathcal{C})-y^i(\mathcal{C}), \tag{117}$$

Applying (117) for $\mathcal{C}=\mathcal{B}$ as well as $\mathcal{C}=\mathcal{B}'$, we obtain

$$g^i(\mathcal{B}')-g^i(\mathcal{B})=p^i(\mathcal{B})-p^i(\mathcal{B}')+\delta^i(\mathcal{B}')-\delta^i(\mathcal{B})-\zeta^i \tag{118}$$

for any bank $i\in\mathcal{D}(\mathcal{B})$. Since any bank $i\in\mathcal{R}$ is rescued in the bailout $s(\mathcal{B})$, we have $g^i(\mathcal{B})=0$ and $p^i(\mathcal{B})=L^i$. Equation (117) thus implies that $g^i(\mathcal{B}')-g^i(\mathcal{B})=p^i(\mathcal{B})-p^i(\mathcal{B}')+\delta^i(\mathcal{B}')-S^i(\mathcal{B}')$ for any $i\in\mathcal{R}$. Together with (116) and (118), this yields

$$\begin{aligned}\sum_{i\in\mathcal{D}(\mathcal{B}')}(g^i(\mathcal{B}')-g^i(\mathcal{B}))=&\sum_{i\in\mathcal{D}(\mathcal{B}')}(p^i(\mathcal{B})-p^i(\mathcal{B}'))+\sum_{i\in\mathcal{R}}(\delta^i(\mathcal{B}')-S^i(\mathcal{B}'))+\sum_{i\in\mathcal{D}(\mathcal{B})}\Big(\min\Big(\beta\zeta^i,\delta^i(\mathcal{B}')\Big)-\zeta^i\Big)\\ =&\sum_{i\in\mathcal{S}(\mathcal{B}')}\zeta^i+\sum_{i\in\mathcal{R}}(\delta^i(\mathcal{B}')-S^i(\mathcal{B}'))+\sum_{i\in\mathcal{D}(\mathcal{B})}\min\Big(\beta\zeta^i,\delta^i(\mathcal{B}')\Big),\end{aligned} \tag{119}$$

where we have used that

$$\sum_{i\in\mathcal{D}(\mathcal{B}')}(p^i(\mathcal{B})-p^i(\mathcal{B}'))=\sum_{i=1}^{n}(p^i(\mathcal{B})-p^i(\mathcal{B}'))=\sum_{i=1}^{n}\zeta^i$$

since π is column-stochastic. It now follows from (115) and (119) that

$$\begin{aligned}W_\lambda(s(\mathcal{B}'))-W_\lambda(s(\mathcal{B}))=&(1+\lambda)\sum_{i\in\mathcal{S}(\mathcal{B})}\zeta^i-\lambda\sum_{i\in\mathcal{S}(\mathcal{B})}\min\Big(\zeta^i,C^i(\mathcal{B})\Big)-(1+\lambda)\sum_{i\in\mathcal{R}}S^i(\mathcal{B}')\\ &+(1+\lambda)\sum_{i\in\mathcal{R}}\delta^i(\mathcal{B}')+(1+\lambda)\sum_{i\in\mathcal{D}(\mathcal{B})}\min\Big(\beta\zeta^i(\mathcal{B}'),\delta^i(\mathcal{B}')\Big)\end{aligned} \tag{120}$$

which readily implies (22). □

Proof of Lemma 5.5. Fix a set $\mathcal{B}_*\in\mathcal{B}_P$ of banks to be rescued in the threatened bailout. Fix two sets of banks $\mathcal{B}'\subseteq\mathcal{B}$ and let $\zeta=\pi(p(\mathcal{B})-p(\mathcal{B}'))$. Observe that $\zeta\geq 0$ since $p(\,\cdot\,)$ is monotonic in the set of rescued banks. It follows in the same way as in the proof of Lemma 5.2 that $C(\mathcal{B}')=\big(C(\mathcal{B})-\zeta\big)^{+}$. Together with the definition of $\eta(\mathcal{B})$ in (23), this implies

$$\eta(\mathcal{B}')=\min\big((\pi(p(\mathcal{B})-p(\mathcal{B}_*))-s(\mathcal{B}_*)-\zeta\big)^{+},C(\mathcal{B}'))=\big(\eta(\mathcal{B})-\zeta\big)^{+}.$$

It now follows from the elementary identity $a-(a-b)^+ = \min(a,b)$ that

$$\eta(\mathcal{B}) - \eta(\mathcal{B}') = \min(\zeta, \eta(\mathcal{B})). \tag{121}$$

Together with (120), this implies the statement. $\square$

Proof of Lemma 5.6. Fix $\mathcal{B}_* \in \mathcal{B}_P$, from which the threats are chosen. Fix two sets $\mathcal{B} \subseteq \mathcal{B}'$ and note that $\eta(\mathcal{B}') \geq \eta(\mathcal{B})$ by (121). It follows that

$$W_\lambda(s(\mathcal{B}')) - \lambda \sum_{j=1}^{m(\mathcal{B})} \eta^{i_j(\mathcal{B})}(\mathcal{B}') \leq W_\lambda(s(\mathcal{B})) - \lambda \sum_{j=1}^{m(\mathcal{B})} \eta^{i_j(\mathcal{B})}(\mathcal{B}) < W_P^*.$$

Because $m'(\mathcal{B})$ is the size of the smallest set of contributors to attain welfare losses lower than W_P^* with contributions $\eta(\mathcal{B}')$, this shows that $m(\mathcal{B}') \leq m(\mathcal{B})$. Since $\eta^i(\mathcal{B}') \geq \eta^i(\mathcal{B})$ for any bank i, the k^{th}-largest element in $\eta(\mathcal{B}')$ must be at least as large as the k^{th}-largest element in $\eta(\mathcal{B})$ for any k. This shows that

$$\begin{aligned} W_{B_*}(\mathcal{B}') &\leq W_P^* - \lambda \eta^{i_{m(\mathcal{B}')+1}(\mathcal{B}')}(\mathcal{B}') \\ &\leq W_P^* - \lambda \eta^{i_{m(\mathcal{B}')+1}(\mathcal{B})}(\mathcal{B}) \leq W_P^* - \lambda \eta^{i_{m(\mathcal{B})+1}(\mathcal{B})}(\mathcal{B}), \end{aligned} \tag{122}$$

where we used $m(\mathcal{B}') \leq m(\mathcal{B})$ in the last inequality. To show $W_{\mathcal{B}_*}(\mathcal{B}') \leq W_{\mathcal{B}_*}(\mathcal{B})$, we distinguish two cases. Consider first the case when $m(\mathcal{B}') = m(\mathcal{B})$. Then

$$W_{\mathcal{B}_*}(\mathcal{B}') \leq W_\lambda(s(\mathcal{B}')) - \lambda \sum_{i=1}^{m(\mathcal{B})} \eta^{i_j(\mathcal{B}')}(\mathcal{B}') \leq W_\lambda(s(\mathcal{B})) - \lambda \sum_{i=1}^{m(\mathcal{B})} \eta^{i_j(\mathcal{B})}(\mathcal{B}).$$

Together with (122), this implies $W_{\mathcal{B}_*}(\mathcal{B}') \leq W_{\mathcal{B}_*}(\mathcal{B})$. If $m(\mathcal{B}') < m(\mathcal{B})$, then

$$\begin{aligned} W_{\mathcal{B}_*}(\mathcal{B}') &\leq W_P^* - \lambda \eta^{i_{m(\mathcal{B}')+1}(\mathcal{B}')}(\mathcal{B}') \\ &\leq W_P^* - \lambda \eta^{i_{m(\mathcal{B})}(\mathcal{B})}(\mathcal{B}) \leq W_\lambda(s(\mathcal{B})) - \lambda \sum_{i=1}^{m(\mathcal{B})} \eta^{i_j(\mathcal{B})}(\mathcal{B}), \end{aligned}$$

where we have used the fact that $W_P^* \leq W_\lambda(s(\mathcal{B})) - \lambda \sum_{i=1}^{m(\mathcal{B})-1} \eta^{i_j(\mathcal{B})}(\mathcal{B})$ by definition of $m(\mathcal{B})$. Together with (122), this shows again that $W_{\mathcal{B}_*}(\mathcal{B}') \leq W_{\mathcal{B}_*}(\mathcal{B})$. $\square$

H.3 Counterexample

Consider a network consisting of $n=4$ banks with assets and liabilities given as in the left panel of Figure 1 and the interbank network given in the right panel of Figure 1. Let further $\gamma = 0, \beta = 0.4$, and $\lambda = 0.5$. Banks 1 and 2 are fundamentally defaulting banks. In absence of intervention, they cannot repay what they owe to banks 3 and 4. This causes the contagious default of bank 3 because it becomes unable to repay its senior creditors. The clearing payment vector is $p_N = (0.4, 0.4, 0, 0)$

Bank	L^i	w^i	c^i
1	2.9	0	1
2	3	0	1
3	0	10	8.5
4	0	0	1

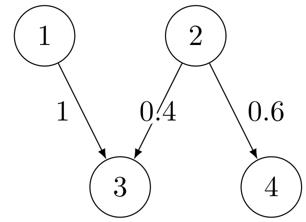

Figure 1: Example of a network with $n = 4$ banks, in which the equilbirium partial bail-in does not rescue all the banks that are bailed out in the optimal partial bailout.

and welfare losses without intervention are equal to 9.824.

A large part of those welfare losses come from the default of bank 3 because of its large asset value. Solvency of bank 3 can be established by either rescuing bank 1 or bank 2. Because the liabilities and hence the shortfall of bank 1 are slightly smaller than those of bank 2, the optimal bailout rescues bank 1 by awarding subsidies $s = (1.9, 0, 0, 0)$. This guarantees solvency of banks 1, 3, and 4 and leads to welfare losses $W_P^* = 1.55$.

For the optimal bail-in, additional considerations are taken into account. The bail-in that provides subsidies only to bank 1 leads to the same clearing payment vector as the optimal bailout. It follows from Lemma 5.3 that the threat level towards all banks is 0 and, thus, that bail-in coincides with the optimal bailout. In comparison, the bail-in that rescues bank 2 provides benefits to bank 4 that id not present in the optimal bailout. Thus, bank 4 has an incentive to participate in the bail-in up to a contribution of size $b^4 = 1.56$. The corresponding bail-in leads to welfare losses of $W_\lambda(\{2\}) = 0.82$. Finally, the bail-in that rescues both banks 1 and 2 allows the regulator to extract contributions from both banks 3 and 4. However, the additional contribution from bank 3 is not large enough to make up for the additional subsidies paid because the large default costs of bank 3 are already prevented by the rescue of bank 2. Indeed, the corresponding welfare losses are equal to $W_\lambda(\{1, 2\}) = 1.03$, which shows that the optimal bail-in may not save banks that are protected in the optimal bailout.

I Data Calibration Procedure

A total of 48 banks participated in the stress test of the EBA. Values of various banks' balance sheet quantities are reported as of end 2017. For each bank, the EBA reports the exposures to other institutions, computed using an internal rating based (IRB) criterion as of end 2017. We take those exposures as a measure of the bank's total claims A on all other banks in the network. Some of the smallest banks reported zero interbanking claims. We omit those from the analysis, leaving us with a total of 36 banks. To estimate interbank liabilities, we first compute the banks' total liabilities as the difference between their total asset holdings and their equity value E, both of which are reported in the EBA data. We then assume that each bank has the same ratio of interbank to total liabilities. We compute this ratio by imposing that the sum of interbank liabilities across all banks equals the sum of interbank claims. Since most quantities in our model depend on c and w only through their difference $c - w$, we set $c = E + \max(E + L - A, 0)$ and $w = E + \max(A - E - L, 0)$

so that the equity value of each bank corresponds to that implied from the EBA data. We can then apply a shock of size $c^i \geq E^i$ to any bank to induce a fundamental default. All raw data and estimated model quantities are summarized in Table 1. We set $\lambda = 1$ and the recovery rate β to 75%. This value is in line with empirical estimates reported by Moody's; see Footnote 11.

Finally, we discuss how to construct the sparse network π_s using an iterative procedure. Starting with $L_{(0)} = L$, we iteratively assign interbank liabilities in the following way: for any $k \geq 0$, let $j_1^{(k)}$ denote the bank with the largest interbank liabilities in $L_{(k)}$ and for any $\ell < n$, let $j_{\ell+1}^{(k)}$ be the bank i in $\{1, \ldots, n\} \setminus \{j_1^{(k)}, \ldots, j_\ell^{(k)}\}$ with the smallest interbank claims $A^i_{(k)} = \sum_{j=1}^n L^{ij}_{(k)}$ that exceeds $L^{j_\ell}_{(k)}$ if such a bank exists and let $j_{\ell+1}^{(k)}$ be the bank i that maximizes $A^i_{(k)}$ otherwise. To generate a sparse network, we assign interbank liabilities of size $\hat{L}^{ij}_{(k)} = \min(L^j_{(k)}, A^i_{(k)})$ if $j = j_\ell^{(k)}$ and $i = j^{(k)}_{\mathrm{mod}(\ell+1,n)}$ for some ℓ and $\hat{L}^{ij}_{(k)} = 0$ otherwise. We proceed to step $k+1$, assigning the remaining liabilities $L_{(k+1)}$ defined by $L^i_{(k+1)} = L^i_{(k)} - \sum_{j=1}^n \hat{L}^{ji}_{(k)}$ for $i = 1, \ldots, n$ until $L_{(k+1)} = 0$. This algorithm generates a network that is the superposition of ring networks—represented by liabilities $\hat{L}_{(k)}$ generated in step k of the algorithm—which guarantees that the resulting network is connected. While this may not be the sparsest possible network, it generates a very sparse network with normalized Gini index of 0.9981 in our application in Section 6; see also Footnote 41.

Bank Name	Total Claims	Equity	Interbank Claims A^i	Interbank Liabilities L^i	Senior Liabilities w^i	Cash holdings c^i
Erste Group Bank AG	147,449	15,368	10,756	12,394	15,368	32,374
Raiffeisen Bank International AG	61,499	9,839	5,416	4,846	9,839	19,108
Belfius Banque SA	127,440	8,141	35,597	11,192	24,404	8,141
KBC Group NV	226,455	16,552	17,128	19,692	16,552	35,668
Danske Bank	273,199	20,302	14,926	23,725	20,302	49,403
OP Financial Group	91,467	9,973	7,299	7,645	9,973	20,292
BNP Paribas	1,012,707	84,417	63,604	87,088	84,417	192,318
Group Credit Mutuel	430,308	45,578	44,606	36,093	45,578	82,643
Groupe BPCE	668,255	59,490	32,956	57,112	59,490	143,135
Groupe Credit Agricole	1,047,925	84,292	97,114	90,404	84,292	161,874
Societe Generale S.A.	642,940	49,514	53,400	55,672	49,514	101,300
Bayerische Landesbank	193,192	9,393	22,731	17,243	9,393	13,298
Commerzbank AG	314,214	25,985	42,564	27,040	25,985	36,446
Deutsche Bank AG	758,140	57,631	58,015	65,719	57,631	122,966
DZ Bank AG	202,301	19,923	35,800	17,110	19,923	21,156
Landsesbank Baden-Wurttemberg	206,824	12,795	57,434	18,203	39,230	12,795
Landsesbank Hessen-Thuringen Girozentrale	122,115	8,180	15,767	10,688	8,180	11,281
Norddeutsche Landesbank - Girozentrale	71,764	6,229	16,037	6,148	9,888	6,229
Allied Irish Banks Group plc	48,157	11.028	10.064	3,484	11,028	15,475
Bank of Ireland Group plc	68,264	7,617	4,537	5,689	7,617	16,386
Intesa Sanpolo S.p.A.	309,144	43,466	36,125	24,924	43,466	75,731
UniCredit S.p.A.	309,144	43,466	36,125	24,924	54,703	101,448
ABN AMRO Group N.V.	367,487	19,618	14,942	32,635	19,618	56,929
Cooperatieve Rabobank U.A.	547,353	37,204	14,461	47,860	37,204	107,807
ING Groep N.V.	780,776	50,325	76,469	68,528	50,325	92,709
Banco Bilbao Vizcaya Argentaria S.A.	276,960	46,980	75,226	21,575	53,650	46,980
Banco de Sabadell S.A.	108,282	11,111	1,559	9,116	11,111	29,779
Banco Santander S.A.	565,109	77,283	36,878	45,765	77,283	163,453
Nordea Bank - group	437,347	28,008	40.127	38,402	28,008	54,291
Skandinaviska Enskilda Banken - group	209,082	13,452	14,944	18,353	13,452	30,313
Svenska Handelsbanken - group	253,639	12,954	7,339	22,580	12,954	41,149
Swedbank - group	202,830	11,356	6,522	17,963	11,356	34,153
Barclays Plc	562,002	60,765	49,797	47,024	60,765	118,757
HSBC Holdings Plc	1,322,909	125,976	117,004	112,291	125,976	247,239
Lloyds Banking Group Plc	590,827	40,948	8,817	51,587	40,948	124,666
The Royal Bank of Scotland Group Plc	490,122	44,577	23,685	41,799	44,577	107,268

Table 1: Results based on data from the 2018 EBA stress test. All numbers are reported in million dollars.